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-USSR-

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(Presented by Academician L.S. Pontryagin)

The present article deals with optimal control processes with limited phase coordinates and includes a derivation of the equations for the extremals of the corresponding problem in the calculus of variations.

Introduction

The Maximum Principle as applied to optimal process theory (see refs. 1-6) makes it possible to find the extremals of the following problems in the calculus of variations.

Let the vector function

$$\vec{f}(\vec{x},u) = (f^1(\vec{x},u),\dots, f^n(\vec{x},u))$$

in variables $\vec{x}$ and u be defined and continuous over the direct product

$$(\vec{x},u) \in X^n \cdot \Omega,\ \vec{x} \in X^n,\ u \in \Omega,$$

where X^n is the n-dimensional phase space of the problem, Ω is an arbitrary Hausdorff topological space (the space of possible values for the control parameter u). The functions

$f^i(\vec{x}, u)$ are assumed to be continuously differentiable for all coordinates of vector $\vec{x}$.

Let the equation of motion of the phase point have the form

$$\dot{\vec{x}} = \vec{f}(\vec{x}, u) \tag{0.1}$$

and X^n contain the two given points $\vec{\xi}_1$, $\vec{\xi}_2$. We are required to select from a class of permissible controls (such as the class of measurable bounded or piecewise-continuous controls with values from Ω) a function $u(t), t_1 \leq t \leq t_2$ such that the corresponding trajectory $\vec{x}(t)$ of equation (0.1) will connect points $\vec{\xi}_1$, $\vec{\xi}_2$.

$$\vec{x}(t_1) = \vec{\xi}_1 \quad , \vec{x}(t_2) = \vec{\xi}_2$$

and the integral

$$\int_{t_1}^{t_2} L(\vec{x}(t), u(t))\, dt \tag{0.2}$$

assume minimum values. (The scalar function $L(\vec{x}, u)$ satisfies the same conditions as the functions $f^i(\vec{x}, u)$.)

The Maximum Principle is stated in Part 1 of Section 3. The space of possible values Ω for the control parameter can in particular be a closed region of the r-dimensional space E^r. The set of possible values for the phase point $\vec{x}$ must correspond to the entire space X^n, since the Maximum Principle ceases to apply in the contrary case.

Nevertheless, the case where the set of possible values for the phase point $\vec{x}$ is a closed region in X^n with a piecewise-smooth boundary is of definite importance in actual application.

For example, the case where the class of permissible controls $u(t) \in \Omega \subset E^r$ consists of continuous piecewise-smooth controls with a modulus-restricted derivative reduces to this problem. And indeed, regarding u in this case as a phase variable and taking as the control parameter the derivative of u, we have instead of equation (0.1) the following system:

$$\dot{\vec{x}} = \vec{f}(\vec{x}, u)$$
$$\dot{u} = v$$

where v is a piecewise-continuous function, while a portion of the phase coordinates forming the vector u does not come

from the region Ω.

The present paper contains the results I obtained in this area in L.S. Pontryagin's seminar on the theory of oscillations and automatic control.

Earlier, in ref. 7, these results were formulated for the special case of optimally rapid action, i.e., where the subintegral function $L(\vec{x}, u)$ of (0.2) $\equiv 1$.

The basic result of this study is formulated in section 4 in the form of a general principle for finding optimal controls and optimal trajectories.

Of the earlier papers on this subject, I should point out those of A.Ya. Lerner (ref. 8) and Ye.A. Rozenman (ref. 9).

I. Statement of the Problem

1. Basic definitions. Let Ω be an arbitrary set in the r-dimensional linear space

$$\mathcal{E}^r = \{u = (u^1, \dots, u^r)\}.$$

By the class of permissible controls we will designate the set of all piecewise-continuous, piecewise-smooth vector function

$$u(t) = (u^1(t), \dots, u^r(t))$$

with first-order discontinuities determined over an arbitrary segment $t_1 \leq t \leq t_2$ of the time axis an taking on values from the set Ω at each moment of time; the functions in this set will be called permissible controls.

If the segments determining two permissible controls do not coincide, we shall consider them different even if one of the segments is contained in the other and the controls coincide over a smaller segment.

Henceforth, we shall assume at all times that the value of control u(t) at the point of discontinuity τ is equal to the left limit:

$$u(\tau) = u(\tau - 0).$$

In the n-dimensional phase space $X^n = \{\vec{x} = (x^1, \dots, x^n)\}$ of the optimal problem formulated below, let there be given a closed region B with a smooth boundary, determined in the neighborhood of the boundary by the inequality

$$g(\vec{x}) = g(x^1, \ldots, x^n) \leq 0$$

where the scalar function $g(\vec{x})$ has continuous partial derivatives in the region of the boundary $g(\vec{x}) = 0$, and the vector

$$\frac{\partial g(\vec{x})}{\partial \vec{x}} = \text{grad}\, g(\vec{x}) = \left(\frac{\partial g}{\partial x^1}, \ldots, \frac{\partial g}{\partial x^n}\right)$$

does not go to zero anywhere on the boundary.

Thus, the boundary of region B is a regular hypersurface of the space X^n with a continuously changing curvature. The practically important case of a region with a piecewise-smooth boundary automatically reduces to this case (see section 3, note 3 to theorem 2).

2. Statement of the problem. Let the real scalar functions $L(\vec{x}, u), f^i(\vec{x}, u), i = 1, \ldots, n$ be continuous and continuously differentiable with respect to all of the coordinates of the vectors $\vec{x}, u$ on the direct product $B^* \Omega^* \supset B\Omega$, where B^* and Ω^* are open sets from X^n, E^r, contianing, respectively, B, Ω.

Let the equation of motion of the representing point $\vec{x} = (x^1, \ldots x^n)$ have the normal form

$$\dot{\vec{x}} = \vec{f}(\vec{x}, u), \tag{1.1}$$

where

$$\vec{f}(\vec{x}, u) = (f^1(\vec{x}, u), \ldots, f^n(\vec{x}, u)). \tag{1.2}$$

If in the right member of equation (1.1) we replace the argument u with a certain permissible control u(t), then (1.1) becomes an n-dimensional vectorial differential equation, and, setting the initial value $\vec{x}(t_1)$, we obtain a single-valued trajectory $\vec{x}(t)$ of the equation, which corresponds to the chosen control u(t) and is determined over a certain time segment.

The basic results of the present study are formulated in theorem 3, section 4 and consist in the determination of a complete system of necessary conditions which must be satisfied by any regular trajectory of equation (1.1) and the corresponding control, which provide the solution to the optimal problem stated below; the concept of a regular

trajectory is defined in sections 2 and 3. These necessary conditions are presented in the form of a system of equations, and for this reason they are naturally called the extremal equations for the optimal problem under consideration.

Formulation of the optimal problem. In the phase space X^n there are given two points $\vec{\xi}_1, \vec{\xi}_2$ belonging to the closed region B. Let us designate by $U(\vec{\xi}_1, \vec{\xi}_2)$ the set of all permissible controls $u(t), t_1 \leq t \leq t_2$ having the property that the trajectory $\vec{x}(t), t_1 \leq t \leq t_2$ of equation (1.1) corresponding to the control $u(t) \in U(\vec{\xi}_1, \vec{\xi}_2)$, $t_1 \leq t \leq t_2$ connects the points $\vec{\xi}_1, \vec{\xi}_2 : \vec{x}(t_1) = \vec{\xi}_1, \vec{x}(t_2) = \vec{\xi}_2$ and lies completely within the closed region B. From the set $U(\vec{\xi}_1, \vec{\xi}_2)$ we are required to select a control $u(t), t_1 \leq t \leq t_2$ to minimize the integral

$$\int_{t_1}^{t_2} L(\vec{x}(t), u(t))\,dt, \qquad (1.3)$$

where $\vec{x}(t), t_1 \leq t \leq t_2$ is the trajectory of (1.1) corresponding to the control $u(t), t_1 \leq t \leq t_2$.

Thus, for any other control $v(t), t_3 \leq t \leq t_4$, from $U(\vec{\xi}_1, \vec{\xi}_2)$ and the corresponding trajectory $\vec{y}(t), t_3 \leq t \leq t_4$ we have:

$$\int_{t_1}^{t_2} L(\vec{x}(t), u(t))\,dt \leq \int_{t_3}^{t_4} L(\vec{y}(t), v(t))\,dt.$$

Any control which satisfies the condition embodied in the problem formulated above will be called an <u>optimal control</u>; the corresponding trajectory is the <u>optimal trajectory</u>.

Note 1. Since the permissible control can have discontinuities, it is obvious that if the control $u(t), t_1 \leq t \leq t_2$ is optimal and has a corresponding optimal trajectory $\vec{x}(t), t_1 \leq t \leq t_2$, then any segment of the optimal trajectory $\vec{x}(t)$ is also optimal for $t_3 \leq t \leq t_4$ where $t_1 \leq t_3 < t_4 \leq t_2$ and corresponds to the optimal control $u(t), t_3 \leq t \leq t_4$.

Note 2. Since the right member of equation (1.1) does not contain an explicit expression for time, the set $U(\vec{\xi}_1, \vec{\xi}_2)$, in addition to the control $u(t), t_1 \leq t \leq t_2$ also contains the control $v(t) = u(t+\tau), t_1 - \tau \leq t \leq t_2 - \tau$, where τ is arbitrary.

If $L(\vec{x}, u) \equiv 1$, then the integral (1.3) is equal to the difference $t_2 - t_1$, i.e., the time of passage from position $\vec{\xi}_1$ to position $\vec{\xi}_2$ is minimized and we have the optimally rapid problem (ref. 7).

Henceforth, the permissible control and corresponding trajectory will always be assumed as determined on one and the same segment of the time axis. For this reason, this segment will be indicated only for the control or the trajectory. All of the solutions to the differential equation will be assumed to be continuous. In the most important formulations, this continuity will sometimes be mentioned explicitly.

3. A second (equivalent) formulation of the basic problem. Let us give a second (equivalent) formulation of our optimal problem which is more convenient for the purposes of subsequent formulations and proofs.

Let us introduce an (n+1)-dimensional phase space X^{n+1}, whose points, in contradistinction to those of X^n, will be designated as follows (without the bar):

$$x=(x^0, x^1, \dots x^n) = (x^0, \vec{x}) \in X^{n+1},$$

where

$$\vec{x} = (x^1, \dots, x^n) \in X^{n+1}$$

The phase space X^n will henceforth always be identified with the subspace $x^0 = 0$ of space X^{n+1}, so that $x = (0, \vec{x}) = \vec{x}$. Any function $F(\vec{x})$ which depends on the argument $\vec{x} \in X^n$ can be considered a function of $x = (x^0, \vec{x})$ determined by the formula

$$F(x) = F(x^0, \vec{x}) = F(\vec{x}).$$

For the sake of symmetry in the subsequent formulas, let us introduce the notation:

$$L(\vec{x}, u) = f^0(\vec{x}, u) = f^0(x, u),$$

$$f(x, u) = f(\vec{x}, u) = (f^0(\vec{x}, u), f^1(x, u), \dots, f^n(\vec{x}, u)) \\ = (f^0(\vec{x}, u)\, \vec{f}(\vec{x}, u)), \qquad (1.4)$$

where the vector $\vec{f}(\vec{x}, u) = (f^1(\vec{x}, u), \dots, f^n(\vec{x}, u))$ is defined by formula (1.2). Let us note the fact that the function $f(x, u)$ is independent of the coordinate x^0, and that its values lie within X^{n+1}.

By G let us denote the product of the closed region by the axis x^0. The region G, just as B, is given in the neighborhood of its boundary by the inequality

$$g(x) = g(\vec{x}) \leq 0,$$

while the boundary -- by the equation

$$g(x) = 0$$

The region G has a regular n-dimensional boundary of continuously variable curvature, and the vector

$$\frac{\partial g(x)}{\partial x} = \text{grad}\, g(x) = \left(\frac{\partial g}{\partial x^0}, \frac{\partial g}{\partial x^1}, \ldots, \frac{\partial g}{\partial x^n}\right) =$$

$$= \left(0, \frac{\partial g}{\partial x^1}, \ldots, \frac{\partial g}{\partial x^n}\right) = \left(0, \text{grad}\, g(\vec{x})\right)$$

does not go to zero anywhere on the boundary.

Let the equation of motion of the phase point $x = (x^0, \vec{x})$ have the form

$$\dot{x} = f(x, u) \qquad (1.5)$$

This equation obviously combines (1.1) and the relationship

$$\dot{x}^0 = \frac{d}{dt}\int_{t_1}^{t} f^0(\vec{x}(\theta), u(\theta))\, d\theta$$

To each control $u(t), t_1 \leq t \leq t_2$, in the set $U(\vec{\xi}_1, \vec{\xi}_2)$ let us append a trajectory

$$x(t) = (x^0(t), \vec{x}(t)), \quad t_1 \leq t \leq t_2,$$

of equation (1.5) with the initial value

$$x(t_1) = (0, \vec{\xi}_1) \qquad (1.6)$$

where $\vec{x}(t), t_1 \leq t \leq t_2$ is the trajectory of equation (1.1) corresponding to the control u(t) with the boundary values

$$\vec{x}(t_1) = \vec{\xi}_1, \quad \vec{x}(t_2) = \vec{\xi}_2,$$

Consequently, the trajectory $\vec{x}(t), t_1 \leq t \leq t_2$ lies wholly within the closed cylindrical region G and

$$x(t_2) = (x^0(t_2), \vec{\xi}_2),$$

where

$$x^0(t_2) = \int_{t_1}^{t_2} f^0(\vec{x}(t), u(t))\, dt.$$

Thus, the end $x(t_2)$ of the trajectory $x(t)$ lies on the direct π from X^{n+1}, passing through point $(0, \vec{\xi}_2)$ and parallel to the axis x^0.

Conversely, if $u(t), t_1 \leq t \leq t_2$ is an arbitrary permissible control with a trajectory $x(t)$ of equation (1.5)

with an initial value (1.6) and end point $x(t_2)$ lying on the direct π, then $u(t) \in U(\xi_1, \xi_2)$.

In view of these consideration, our problem may be formulated in the following manner.

A second (equivalent) formulation of the optimal problem. We are to find a permissible control $u(t)$, $t_1 \leq t \leq t_2$ such that the end-point $x(t_2)$ of the trajectory of equation (1.5) with the initial value (1.6) lies on the direct π and the coordinate $x^0(t_2)$ is minimum, while the trajectory $x(t)$ lies wholly within the closed region G.

Such a trajectory will also be called optimal.

2. Optimal Trajectories Lying on the Boundary of a Region

This paragraph catains a proof of theorem 1, which gives a complete system of necessary conditions satisfied by any regular optimal trajectory lying wholly on the boundary $g(x)=0$ of region G. The concept of a regular trajectory lying on the boundary of region G is defined in paragraph 1.

1. Basic definitions. Let $\psi = (\psi_0, \psi_1, \cdots, \psi_n)$ be a covariant vector of the space X^{n+1}. The basic role is henceforth played by the scalar function $H(\psi, x, u)$ of the three vectorial arguments ψ, x, u, defined as the scalar product

$$H(\psi, x, u) = \psi \cdot f(x, u) = \sum_{\alpha=0}^{n} \psi_\alpha f^\alpha(x, u) \tag{2,1}$$

where the vector $f(x,u)$ is given by formula (1.4). (Scalar multiplication will henceforth be indicated by means of a point.)

If we hold fixed the arguments ψ, x and vary u over the set Ω, H becomes a function of the single argument u. The precise upper bound of this function (in the set Ω) will be denoted by $M(\psi, x)$:

$$M(\psi, x) = \sup_{u \in \Omega} H(\psi, x, u). \tag{3,2}$$

Let us introduce the notation:

$$\left.\begin{aligned}
\frac{\partial H}{\partial \psi} &= \left(\frac{\partial H}{\partial \psi_0}, \frac{\partial H}{\partial \psi_1}, \ldots, \frac{\partial H}{\partial \psi_n}\right) = \left(f^0(x,u), \ldots, f^n(x,u)\right) = f(x,u), \\
\frac{\partial H}{\partial x} &= \left(\frac{\partial H}{\partial x^0}, \frac{\partial H}{\partial x^1}, \ldots, \frac{\partial H}{\partial x^n}\right) = \left(0, \frac{\partial H}{\partial x^1}, \ldots, \frac{\partial H}{\partial x^n}\right), \\
\frac{\partial H}{\partial u} &= \left(\frac{\partial H}{\partial u^1}, \ldots, \frac{\partial H}{\partial u^r}\right).
\end{aligned}\right\} \quad (2.3)$$

Equation (1.5) can now be written in the form

$$\dot{x} = \frac{\partial H(\psi, x, u)}{\partial \psi} \quad (2.4)$$

Let us consider the linear homogeneous equation with respect to ψ :

$$\dot{\psi} = -\frac{\partial H(\psi, x, u)}{\partial x} \quad (2.5)$$

Equations (2.4)-(2.5) jointly form a Hamiltonian system with Hamiltonian function (2.1). If $\psi(t), x(t), u(t)$ is an arbitrary solution of equation (2.5), where $\psi(t) = (\psi_0(t), \ldots, \psi_n(t))$, then ψ_0 =const, since H is independent of x^0 (see (2.3)).

By analogy with (2.3) let us introduce the definitions:

$$\left.\begin{aligned}
p(x,u) &= \sum_{\alpha=0}^{n} \frac{\partial g(x)}{\partial x^\alpha} f(x,u) = \sum_{\alpha=1}^{n} \frac{\partial g(x)}{\partial x^\alpha} f^\alpha(x,u) = \\
&= \frac{\partial g(x)}{\partial x} \cdot f(x,u), \\
\frac{\partial p(x,u)}{\partial x} &= \left(\frac{\partial p}{\partial x^0}, \frac{\partial p}{\partial x^1}, \ldots, \frac{\partial p}{\partial x^n}\right) = \left(0, \frac{\partial p}{\partial x^1}, \ldots, \frac{\partial p}{\partial x^n}\right), \\
\frac{\partial p(x,u)}{\partial u} &= \left(\frac{\partial p}{\partial u^1}, \ldots, \frac{\partial p}{\partial u^r}\right),
\end{aligned}\right\} \quad (2.6)$$

where $g(x) = 0$ is the equation of the boundary of region G (see section 1, paragraph 3).

In order for trajectory $x(t), t_1 \le t \le t_2$ of equation (1.5), corresponding to $u(t)$, to lie wholly on the boundary $g(x)=0$, it is necessary and sufficient that

$$p(x(t), u(t)) = 0, \quad t_1 \le t \le t_2, \quad g(x(t_1)) = 0.$$

The point $x_1 \in X^{n+1}$ will be called <u>regular relative to point</u> $u_1 \in \Omega$, provided that the following conditions are satisfied:

1. $p(x_1, u_1) = 0$;

2. $\frac{\partial p(x_1, u_1)}{\partial u} \neq 0$;

3. if u_1 is a boundary point of the set Ω, then it is possible to find continuously differentiable scalar functions

$$q_i(u), \quad i = 1, \ldots, s \tag{2.7}$$

such that the set Ω in the neighborhood of point u_1 is given by the system of inequalities

$$q_1(u) \leq 0, \ldots, q_s(u) \leq 0$$

while in point u_1 itself

$$q_1(u_1) = \ldots = q_s(u_1) = 0 \tag{2.8}$$

and the vectors

$$\frac{\partial p(x_1, u_1)}{\partial u}, \frac{\partial q_1(u_1)}{\partial u} = \operatorname{grad} q_1(u_1), \ldots, \frac{\partial q_s(u_1)}{\partial u} = \operatorname{grad} q_s(u_1) \tag{2.9}$$

are independent.

Condition 2 may be considered a special case of condition 3 if we set $s = 0$ in the latter when u_1 is an internal point of the set Ω. This we will do henceforth.

The geometrical significance of condition 3 consists in the fact that in the neighborhood of point u_1 the set Ω is a closed region with a piecewise-smooth boundary, while the (r-1)-dimensional bounds of the region are smooth hypersurfaces of the space $\mathcal{E}^r$, located in general position at the point of intersection of u_1; point u_1 itself lies on the (r-s)-dimensional smooth "ridge" of the boundary, defined as the set of solutions to the system

$$q_1(u) = \ldots = q_s(u) = 0 \tag{2.10}$$

lying near point u_1. This "ridge" is located in the general vicinity of point u_1 with a smooth (r-1)-dimensional hypersurface, given in the neighborhood of point u_1 by the

equation $p(x_1, u_1) = 0$. [See Note at bottom of page 12.]

Note. Using the concept of regularity, we shall now describe a simple construction which will be needed in paragraph 4.

Let $R_1(x, u, \mu)$ be a continuously differentiable scalar function of the scalar arguments x, u and the scalar parameter μ, and let $R(x, u, 0) = p(x, u)$.

If the point x_1 is regular relative to $u_1 \in \Omega$, then the system

$$R_1(x, u, \mu) = q_1(u) = \ldots = q_s(u) = 0$$

is resolvable relative to certain $s+1$ coordinates of the vector u near point $(x_1, u_1, \mu = 0)$, for example, relative to the first $s+1$ coordinates:

$$u^i = u^i(x, u^{s+2}, \ldots, u^r, \mu), \quad i = 1, \ldots, s+1, \quad 1 \leq s+1 \leq r$$

where the functions $u^i, i = 1, \ldots, s+1$ are continuously differentiable with respect to all arguments. Substituting these functions in (1.5) in place of parameters $u^1, \ldots, u^{s+1}$, we obtain:

$$\dot{x} = f(x, u^1(x, u^{s+2}, \ldots, u^r, \mu), \ldots, u^r) = f_1(x, u^{s+2}, \ldots, u^r, \mu). \qquad (2.11)$$

Let $u^{s+2}(t), \ldots, u^r(t), \tau_1 \leq t \leq \tau_2$ be piecewise-continuous and piecewise-smooth functions with values lying near the corresponding coordinates $u_1^{s+2}, \ldots, u_1^r$ of vector u_1. Let us substitute them into (2.11) and denote by $x(t) = x(t, \mu)$ the solution to the resulting equation with the initial value

$$x(\tau_1, \mu) = x_1.$$

Obviously, the solution $x(t)$ is now determined over the entire segment $\tau_1 \leq t \leq \tau_2$ and for a sufficiently small length of segment $\tau_2 - \tau_1$ is the trajectory of equation (1.5) which corresponds to the permissible control

$$u(t) = (u^1(x(t), u^{s+2}(t), \ldots, u^r(t), \mu), \ldots, u^r(t)), \quad \tau_1 \leq t \leq \tau_2$$

and, in addtion, satisfies the equation

$$R(x(t), \mu) = 0, \quad \tau_1 \leq t \leq \tau_2.$$

The values of u(t) lie near u_1, the values $x(t)$ -- near x_1.

If the point x_1 lies on the boundary $g(x) = 0$, the trajectory $x(t, \mu), \tau_1 \leq t \leq \tau_2$ also lies on it, since

$$P(x(t,0)) = R(x(t,0), u(t), 0) = 0$$

Let us denote by $\omega(x)$ the set of those $u \in \Omega$ relative to which the point x is regular. The set $\omega(x)$ can turn out to be empty.

The trajectory $x(t), t_1 \leq t \leq t_2$ of equation (1.5) corresponding to the control u(t) and lying wholly on the boundary of region G will be called regular if $u(t) \in \omega(x(t))$ at each point of continuity t of the control u(t); if, on the other hand, t is a point of discontinuity in the control, then it is required that

$$u(t \pm 0) \in \omega(x(t)).$$

Any optimal trajectory lying on the boundary $g(x) = 0$ and at the same time regular will be called a regular optimal trajectory.

For points x lying on the boundary $g(x)=0$ of region G for which $\omega(x)$ is not empty, let us define the expression $m(\psi, x)$ by the equation

$$m(\psi, x) = \sup_{u \in \omega(x)} H(\psi, x, u). \qquad (2.12)$$

If x is a regular point in the boundary $g(x)=0$ relative to the point $u \in \Omega$ and in the point (ψ, x, u) where vector ψ is arbitrary,

$$H(\psi, x, u) = m(\psi, x),$$

then according to the rule for Lagrange multipliers, there exist real numbers $\lambda, \nu_1, \ldots, \nu_s$ such that

$$\frac{\partial H(\psi, x, u)}{\partial u} = \lambda \frac{\partial p(x, u)}{\partial u} + \sum_{\alpha=1}^{s} \nu_\alpha \frac{\partial q_\alpha(u)}{\partial u} \qquad (2.13)$$

where $q_i(u), i=1,\ldots,s$ are the functions (2.7) which play a part in the definition of the concept of regularity. Generally speaking, all of the ν_i do not simultaneously go to zero if u is a boundary point of the set $\Omega(s>0)$. If u is an internal point of set $\Omega(s=0)$, it then follows from (2.13) that the vector $\frac{\partial H(\psi, x, u)}{\partial u}$ is collinear to the vector $\frac{\partial p(x, u)}{\partial u}$.

[Note] Although the functions (2.7) are not determined for single values, it follows from condition 3 that the (r-s)-dimensional "ridge" (2.10) and the number s do take on single values by means of point u_1.

2. THEOREM 1. Let $x(t)$, $t_1 \leq t \leq t_2$ be the regular optimal trajectory of equation (1.5) corresponding to the optimal control u(t) and lying wholly on the boundary of region G. Then there will be found a continuous non-zero covariant vector function $\psi(t) = (\psi_0(t), \ldots, \psi_n(t))$, $t_1 \leq t \leq t_2$ and a piecewise-continuous, piecewise smooth scalar function $\lambda(t)$, $t_1 \leq t \leq t_2$ such that the following equations will be satisfied on the segment $t_1 \leq t \leq t_2$:

$$\dot{x} = \frac{\partial H(\psi, x, u)}{\partial \psi} = f(x, u), \qquad (2.14)$$

$$\dot{\psi} = -\frac{\partial H(\psi, x, u)}{\partial x} + \lambda(t) \frac{\partial p(x, u)}{\partial x}, \qquad (2.15)$$

$$H(\psi(t), x(t), u(t)) = M(\psi(t), x(t)) = 0 \qquad (2.16)$$

along with the conditions:

a) coordinate $\psi_0(t)$ = const 0;

b) vector $\psi(t_1)$ is not collinear to the vector of the normal $\operatorname{grad} g(x(t_1))$ to the boundary $g(x) = 0$ at point $x(t_1)$;

c) at all points at which the function is differentiable, the vector

$$\frac{d\lambda(t)}{dt} \operatorname{grad} g(x(t))$$

is directed into region G or goes to zero; the function $\lambda(t)$ is determined from maximum condition (2.16) as a Lagrange multiplier for the vector $\frac{\partial p(x, u)}{\partial u}$ in formula (2.13).

We now proceed to make several fundamental remarks regarding the above theorem.

Note 1. The equality $\psi_0(t) = \text{const}$ is a consequence of the independence of the right member of equation (2.15) of x^0.

Equations (2.14)-(2.16) and condition a) are analogous to the Maximum Principle. Conditions b) and c) specific to the present case and are discussed in notes 4 and 5.

Note 2. Let us examine more closely the method of determining $\lambda(t)$ as a Lagrange multiplier in formula (2.13).

In theorem 1 it is stated that it is possible to break up the segment $t_1 \leq t \leq t_2$ in such a manner that for each partial segment it will be possible to find $s > 0$ $q_i(u)$, $i = 1, \ldots, s$ differentiated functions and s piecewise-continuous, piecewise-smooth functions $v_i(t)$, $i = 1, \ldots, s$ determined on the partial segment under consideration and satisfying the following equation on this segment:

$$\lambda(t) = \sum_{\alpha=0}^{n} \psi_\alpha(t) a^\alpha(t) = \psi(t)\cdot \mathbf{a}(t)$$

where $\mathbf{a}(t)=(a^0(t),\dots,a^n(t))$ is a piecewise-continuous, piecewise-smooth contravariant vector function. Substituting this expression for $\lambda(t)$ into (2.15), we obtain a homogeneous linear equation relative to $\psi(t)$, for which the singularity theorem holds; this is the source of our statement.

Note 4. To find out the meaning of b) let us note that system (2.14)-(2.18) has the following trivial solution:

$$u(t), x(t), \psi(t) = \mu \operatorname{grad} g(x(t)), \lambda(t) = \mu, \quad t_1 \le t \le t_2,$$

where μ is an arbitrary number. This is easily demonstrated by direct substitution.

It is also easy to check the truth of the statement that if

$$u(t), x(t), \psi(t), \lambda(t)$$

is some solution of system (2.14)-(2.18),

$$u(t), x(t), \psi(t) + \mu \operatorname{grad} g(x(t)), \lambda(t) + \mu$$

is also a solution where μ is an arbitrary number.

Taking into account note 3, it is possible to assert that if

$$\psi(t_1) = \mu \operatorname{grad} g(x(t_1)),$$

then

$$\psi(t) \equiv \mu \operatorname{grad} g(x(t)), \quad t_1 \le t \le t_2,$$

i.e., $\psi(t)$ is a trivial solution.

Note 5. Condition c) arises as a result of the fact that in the derivation of the necessary conditions, the optimal trajectory lying on the boundary $g(x)=0$ is compared not only with neighboring trajectories also lying on the boundary, but also with all neighboring trajectories in the closed region G.

The proof of theorem 1 requires certain notions whose description takes up the following two subsections.

3. Certain basic notions. By analogy with (2.3) and (2.6) let us introduce the definition

$$\frac{\partial f}{\partial u} = \left\| \frac{\partial f^i}{\partial u^j} \right\|, \quad i = 0, 1, \dots, n, \; j = 1, \dots, r$$

and regard this matrix as an operator from the space of contravariant vectors $u = (u^1, \dots, u^r) \in E^r$ into the space of contravariant vectors $\chi = (\chi^0, \chi^1, \dots, \chi^n) \in X^{n+1}$, and at the same time as an operator from the covariant vector space $\psi = (\psi_0, \dots, \psi_n)$ of space X^{n+1} into the covariant vector space of space E^r:

$$\frac{\partial f}{\partial u} v = \left(\sum_{j=1}^{r} \frac{\partial f^0}{\partial u^j} v^j, \dots, \sum_{j=1}^{r} \frac{\partial f^n}{\partial u^j} v^j \right),$$
$$\psi \frac{\partial f(x,u)}{\partial u} = \left(\sum_{\alpha=0}^{n} \psi_\alpha \frac{\partial f^\alpha}{\partial u^1}, \dots, \sum_{\alpha=0}^{n} \psi_\alpha \frac{\partial f^\alpha}{\partial u^r} \right) = \frac{\partial H(\psi, x, u)}{\partial u} \qquad (2.17)$$

Let $\Lambda = (\Lambda^0, \Lambda^1, \dots, \Lambda^n)$ be a contravariant vector from X^{n+1}. The matrices

$$\Lambda \frac{\partial^2 f(x,u)}{\partial x^2} = \left\| \Lambda^\gamma \frac{\partial^2 f^\alpha(x,u)}{\partial x^\beta \partial x^\gamma} \right\|, \quad \alpha, \beta = 0, \dots, n,$$

$$\Lambda \frac{\partial^2 f(x,u)}{\partial x \partial u} = \left\| \Lambda^\gamma \frac{\partial^2 f^\alpha(x,u)}{\partial x^\gamma \partial u^j} \right\|, \quad \alpha = 0, \dots, n, \; j = 1, \dots, r$$

will also be looked upon as operators acting on vectors ψ, χ, v according to the formulas:

$$\left.\begin{aligned} \left(\Lambda \frac{\partial^2 f}{\partial x^2}\right) \psi &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} \Lambda \cdot \psi\right) = \sum_{\alpha, \gamma = 0}^{n} \psi_\alpha \frac{\partial^2 f^\alpha}{\partial x \partial x^\gamma} \Lambda^\gamma, \\ \left(\Lambda \frac{\partial^2 f}{\partial x^2}\right) \chi &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} \cdot \chi\right) \Lambda = \sum_{\beta, \gamma} \frac{\partial^2 f}{\partial x^\beta \partial x^\gamma} \chi^\beta \Lambda^\gamma, \\ \left(\Lambda \frac{\partial^2 f}{\partial x \partial u}\right) v &= \Lambda \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u} v\right) = \sum_{\gamma, j} \frac{\partial^2 f}{\partial x^\gamma \partial u^j} \Lambda^\gamma v^j \end{aligned}\right\} \qquad (2.18)$$

In the two following subsection $x(t)$, $t_0 \le t \le T$ is an arbitrary regular trajectory of the equation (1.5) corresponding to the control u(t) and lying on the boundary $g(x) = 0$ of region G. In paragraph 5 we shall assume that u(t) and $x(t)$ are optimal.

We now proceed to construct scalar function $K(x, t, u)$ (formula (2.21)), which plays a fundamental role in the proof of theorem 1.

A) Construction of function $K(x, t, u)$. Let us fix $s \ge 0$ points $\xi_i, i = 1, \dots, s$ on trajectory $x(t)$, none of which coincides with the end of trajectory $x(t)$; the equalities $\xi_i = x(t_0)$ and $\xi_i = \xi_j$ are possible for i ≠ j. By N_i let us denote a vector not tangential to boundary $g(x) = 0$ at the point ξ_i and directed out from G; otherwise, vector N_i is arbitrary. Since the region G is defined by the inequality $g(x) \le 0$,

$$\operatorname{grad} g(\xi_i) \cdot N_i > 0.$$

Let [illegible] be an arbitrary sufficiently small neighborhood of the point [illegible] in [illegible], whose closure does not contain the end of [illegible]; the only additional requirement placed on [illegible] is that for [illegible] the closure of neighborhood [illegible] should not contain point [illegible]. The "sufficient smallness" of the neighborhood is characterized by the fact that

$$[illegible]$$

By [illegible] let us denote the differentiable scalar which satisfies the following conditions: [illegible] in a certain neighborhood of point [illegible] contained in [illegible], [illegible] for [illegible], and [illegible] outside of [illegible].

Let us introduce the scalar function

$$[illegible] \qquad (2.19)$$

where [illegible] is a scalar parameter.

Function [illegible] depends on elements [illegible] which take part in its construction; for this reason we shall speak of functions of the form (2.19).

Obviously, if [illegible] is sufficiently small and the point x which lies near the boundary of region G satisfies the equation

$$[illegible]$$

If, in addition to this, [illegible] does not belong to the combination of neighborhoods [illegible] or [illegible], then [illegible] lies on the boundary [illegible].

Let [illegible] be functions of the form (2.19), [illegible] -- elements which take part in the construction of the function [illegible] -- elements taking part in the construction of the function [illegible].

By

$$[illegible]$$

let us designate a function of the form (2.19) defined by the equation

$$[illegible] \qquad (2.20)$$

The function [illegible] will be defined by the equation

$$[illegible] \qquad (2.21)$$

For $\mu = 0$ it follows from (2,8) that

$$R(x, u, 0) = \frac{\partial g(x)}{\partial x} \cdot f(x, u) = p(x, u). \qquad (2,22)$$

In accordance with (2,20) we define

$$R_1(x, u, \mu) \oplus R_2(x, u, \mu) = \frac{\partial}{\partial x}[h(x, u, \mu) \oplus h_2(x, u, \mu)] f(x, u) \qquad (2,23)$$

Henceforth, the parameter μ will assume values close to zero. For this reason, instead of μ we shall write $\varepsilon\,\delta\mu$, where ε will hence in all cases stand for a positive infinitesimally small value. The case when ε can equal zero will be considered separately. By $o(\varepsilon)$ let us denote a magnitude of a higher order than ε :

$$\frac{o(\varepsilon)}{\varepsilon} \to 0 \text{ for } \varepsilon \to 0.$$

Let the functions $v(t), y(t), t_1 \leq t \leq t_2$ where $v(t)$ is a permissible control, $y(t)$ is a permissible function satisfy the system

$$\dot{y} = f(y, v)$$

$$R(y, v, \varepsilon\,\delta\mu) = 0. \qquad (2,24)$$

If $\varepsilon\,\delta\mu \geq 0$, and $y(t)$ lies sufficiently close to $x(t)$ and $h(y(t_1), \varepsilon\delta\mu) = 0$, then $y(t) \in G$ for any t, since

$$\frac{d h(y(t), \varepsilon\delta\mu)}{dt} = R(y(t), v(t), \varepsilon\delta\mu) = 0.$$

The control u(t) and trajectory $x(t)$ satisfy the system (2,24) for $\delta\mu = 0$.

In proving theorem 1, trajectory $x(t)$ should be varied in such a way that the varied trajectory does not emerge from region G. For this purpose we shall now introduce the concept of an equation in variations for system (2,24).

Let us note that formula (2,24), denotes not a single system, but rather an entire family of systems which depend on the choice of function R of the form (2,21).

B) The equation in variations for system (2,24).

Lemma 1. Ther exists a piecewise-continuous, piecewise smooth contravariant vector function

$$\Lambda(t) = (\lambda^0(t), \ldots, \lambda^n(t)), \quad t_1 \leq t \leq t_2$$

which depends only on functions $u(t), x(t)$ (and hence independent of the form of function R) such that for any vector $\delta x_1 \in X^{n+1}$ and any value of the parameter $\delta\mu$ it is possible to construct a solution $\vartheta(t), y(t), t_1 \le t \le t_2$ to the system (2.24) which satisfies the initial value

$$y(t_1) = x(t_1) + \varepsilon \delta x_1 + o(\varepsilon) \tag{2.25}$$

and is represented on the segment $t_1 \le t \le t_2$ in the form

$$\vartheta(t) = u(t) + \varepsilon \delta u(t) + o(\varepsilon) \tag{2.26}$$

$$y(t) = x(t) + \varepsilon \delta x(t) + o(\varepsilon) \tag{2.27}$$

where the piecewise-continuous, continuous-smooth function $\delta u(t)$ and the continuous function $\delta x(t)$ are independent of ε and, in addition, function $\delta u(t)$ satisfies the equation

$$\left(\frac{\partial f(x(t), u(t))}{\partial u} + \Lambda(t) \frac{\partial \varphi(x(t), u(t))}{\partial u}\right) \delta u(t) = 0 \tag{2.28}$$

on the segment $t_1 \le t \le t_2$.

Proof. The segment $t_1 \le t \le t_2$ will be subdivided by the points

$$t_1 = \tau_0 < \tau_1 < \ldots < \tau_k < \tau_{k+1} = t_2$$

into partial segments $\tau_i \le t \le \tau_{i+1}, i = 0, 1, \ldots, k$, of sufficiently small length. Let us select the points τ_i in such a manner that they include all of the points of discontinuity of the control u(t) and its derivative over the segment $t_1 \le t \le t_2$. The "sufficient smallness" of the partial segment lengths is characterized by the requirement that all of the constructions described below be satisfied. From the regularity of trajectory $x(t)$ it follows directly that for given $\delta x_1, \delta\mu$ and sufficiently small ε such a choice of division points (not single-valued, of course) is always possible.

Let us assume that the solution $\vartheta(t), y(t)$ of system (2.24) representable in the form (2.26)-(2.27) and satisfying the initial condition (2.25), is already constructed on the segment $t_1 \le t \le \tau_i, i \geq 0$. Let us extend this solution onto the segment $\tau_i \le t \le \tau_{i+1}$ preserving the continuity of trajectory $y(t)$ and the properties expressed in the equalities (2.26) and (2.27).

The point $x(\tau_i)$ is regular relative to the point $+0$). The corresponding functions (2.7) which play a [illegible]t in the definition of regularity will be designated as

$$q_i(u), \quad i=1,\dots,s \quad (s \geqslant 0).$$

As a result of (2.22) and the smallness of the [illegible]ment $\tau_i \le t \le \tau_{i+1}$ the vectors

$$\frac{\partial R(x(t),u(t),\varepsilon\delta\mu)}{\partial u}, \frac{\partial q_1(u(t))}{\partial u}, \dots, \frac{\partial q_s(u(t))}{\partial u}, \tau_i \le t \le \tau_{i+1},$$

[illegible] independent (for sufficiently small ε). For example,

$$\begin{vmatrix} \frac{\partial R}{\partial u^1}, \frac{\partial q_1}{\partial u^1}, \dots, \frac{\partial q_s}{\partial u^1} \\ \dots\dots\dots\dots \\ \frac{\partial R}{\partial u^{s+1}}, \frac{\partial q_1}{\partial u^{s+1}}, \dots, \frac{\partial q_s}{\partial u^{s+1}} \end{vmatrix} \neq 0 \qquad (2.[illegible])$$

Then in the neighborhood of point $(\tau_i, u(\tau_i), x(\tau_i), \varepsilon\delta\mu=0)$ [illegible] system

$$R(y,v,\varepsilon\delta\mu) = \sigma_1(v,t) = \dots = \sigma_s(v,t) = 0, \qquad (2.30)$$

[illegible]re

$$\sigma_\alpha(v,t) = q_\alpha(v) - q_\alpha(u(t)), \quad \alpha=1,\dots,s.$$

[illegible] a single-valued solution relative to s+1 variables $\dots, v^{s+1}$:

$$v^\alpha = v^\alpha(y, v^{s+2}, \dots, v^r, \varepsilon\delta\mu, t), \alpha=1,\dots,s+1, \qquad (2.31)$$

[illegible]re v^α are continuously differentiable functions.

Substituting in function (2.31) in place of $v^{s+2}, \dots, v^r$ [illegible]pectively, $u^{s+2}(t), \dots, u^r(t)$, we obtain s+1 functions $(y,\varepsilon\delta\mu,t), \alpha=1,\dots,s+1$.

[illegible] us define the vector function $v(y,\varepsilon\delta\mu,t)$ by means of [illegible] equation

$$(y,\varepsilon\delta\mu,t) = (v^1(y,\varepsilon\delta\mu,t),\dots,v^{s+1}(y,\varepsilon\delta\mu,t),u^{s+2}(t),\dots,u^r(t)) \quad (2.32)$$

[illegible] substitute in the equation

$$\dot{y} = f(y,v)$$

place of v in function (2.32). Taking the solution of the

resultant differential equation on the segment $\tau_i \le t \le \tau_{i+1}$ with the already available value $y(\tau_i)$, we obtain the required extension of the solution $y(t)$.

The extension of the function v(t) onto the semi-interval $\tau_i < t \le \tau_{i+1}$ is represented by the function

$$v(t) = v(y(t), \varepsilon\delta_\mu, t), \ \tau_i < t \le \tau_{i+1} \qquad (2.33)$$

which is obtained by the substitution of $y(t), \tau_i < t \le \tau_{i+1}$ in place of y in (2.32).

The properties expressed in equations (2.26–2.27) are checked directly for the functions $v(t), y(t), \tau_i \le t \le \tau_{i+1}$. The permissibility of the control v(t) follows from the equalities (2.30); in actuality, for any $\alpha = 1, \ldots, s$ we have:

$$\sigma_\alpha(v(t), t) = q_\alpha(v(t)) - q_\alpha(u(t)) = 0,$$

i.e.,

$$q_\alpha(v(t)) = q_\alpha(u(t)) \le 0 \quad \text{for } \tau_i < t \le \tau_{i+1}$$

From our construction, we have, in addition, the following fact. Independently of the initial value (2.25) and the magnitude $\delta\mu$, the extension (2.33) on the semi-interval $\tau_i < t < \tau_{i+1}$ has the form

$$v(t) = u(t) + \varepsilon\delta u(t) + o(\varepsilon),$$

where

$$\delta u(t) = (\delta u^1(t), \ldots, \delta u^{s+1}(t), 0, \ldots, 0). \qquad (2.34)$$

It remains to determine the function $\Lambda(t)$ and to prove the equality (2.28). We will assume the function $\Lambda(t)$ to be determined on the segment $t_1 \le t \le \tau_i$ and shall determine it on the semiinterval $\tau_i < t \le \tau_{i+1}$.

For every j, $j = 0, \ldots, n$, let us determine on the semiinterval s+1 continuous smooth functions $\lambda^j(t), \ell^j_\beta(t), \beta = 1, \ldots, s$ as the solution of the linear system

$$\frac{\partial f^j(x(t), u(t))}{\partial u^\alpha} + \lambda^j(t)\frac{\partial p(x(t), u(t))}{\partial u^\alpha} + \sum_{\beta=1}^{s} \ell^j_\beta \frac{\partial q_\beta(u(t))}{\partial u^\alpha} = 0$$

$$(\alpha = 1, \ldots, s+1), \qquad (2.35)$$

which is solvable, since its determinant coincides with the determinant (2.29) for $\varepsilon\delta\mu = 0$. Summing these equations over α with coordinates $\delta u^\alpha(t)$ of vector (2.34), we obtain:

$$\sum_{\alpha=1}^{r}\left(\frac{\partial F^j}{\partial u^\alpha}+\lambda^j(t)\frac{\partial p}{\partial u^\alpha}+\sum_{\beta=1}^{s} l^j_\beta(t)\frac{\partial q_\beta}{\partial u^\alpha}\right)\delta u^\alpha(t)=$$

$$=\left(\frac{\partial F^j}{\partial u}+\lambda^j(t)\frac{\partial p}{\partial u}+\sum_{\beta=1}^{s} l^j_\beta(t)\frac{\partial q_\beta}{\partial u}\right)\delta u(t)=0,\quad j=0,\dots,n. \tag{2.36}$$

If we define the contravariant vector functions $\Lambda(t), L_\beta(t)$ on the semiinterval $\tau_i < t \leq \tau_{i+1}$ by means of the equations

$$\Lambda(t)=(\lambda^0(t),\dots,\lambda^n(t)),$$

$$L_\beta(t)=(l^0_\beta(t),\dots,l^n_\beta(t)),$$

the relationship (2.36) can be rewritten in the form (see formulas (2.18)):

$$\left(\frac{\partial f}{\partial u}+\Lambda(t)\frac{\partial p}{\partial u}+\sum_{\alpha=1}^{s} L_\alpha(t)\frac{\partial q_\alpha}{\partial u}\right)\delta u(t)=0 \tag{2.37}$$

As a result of (2.30) the function $v(t)=u(t)+\varepsilon\delta u(t)+o(\varepsilon)$ satisfies the equations

$$\sigma_\alpha(v(t),t)=\sigma_\alpha(u(t)+\varepsilon\delta u(t)+o(\varepsilon),t)=$$

$$=\sigma_\alpha(u(t),t)+\varepsilon\frac{\partial\sigma_\alpha(u(t),t)}{\partial v}\delta u(t)+o(\varepsilon)=0,\ \alpha=1,\dots,s.$$

But

$$\sigma_\alpha(u(t),t)=q_\alpha(u(t))-q_\alpha(u(t))=0$$

$$\frac{\partial\sigma_\alpha(u(t),t)}{\partial v}=\frac{\partial q_\alpha(u(t))}{\partial u},$$

consequently $\dfrac{\partial q_\alpha(u(t))}{\partial u}\delta u(t)=0,\quad \alpha=1,\dots,s$

and, hence, the equation (2.37) is equivalent to

$$\left(\frac{\partial f}{\partial u}+\Lambda(t)\frac{\partial p}{\partial u}\right)\delta u(t)=0$$

which coincides with (2.28).

Thus, lemma 1 has been completely proved.

Substituting expressions (2.26) and (2.27) in system (2.24) and equating the members for ε, we obtain:

$$\delta\dot{x}=\frac{\partial f(x(t),u(t))}{\partial x}\delta x+\frac{\partial f(x(t),u(t))}{\partial u}\delta u,$$

$$\frac{\partial p(x(t),u(t))}{\partial x}\delta x+\frac{\partial p(x(t),u(t))}{\partial u}\delta u+\frac{\partial R(x(t),u(t),0)}{\partial\mu}=0$$

Multiplying the second equation by $\Lambda(t)$, adding it to the first, and taking into account the equality (2.23), we obtain:

$$\delta \dot{x} = \left(\frac{\partial f(x(t),u(t))}{\partial x} + \Lambda \frac{\partial p(x(t),u(t))}{\partial x}\right) + \Lambda(t)\frac{\partial R(x(t),u(t),0)}{\partial \mu}\delta\mu \qquad (2.38)$$

The resultant linear non-homogeneous equation relative to δx will be called the <u>equation in variations for system</u> (2.24). It is independent of $\delta u(t)$ (which fact is of extreme importance for what follows!) and is satisfied by the main part of the increment of the varied trajectory constructed by the aforementioned standard method (with respect to the given vector δx_1 and the given value $\delta\mu$).

Function $\delta x(t), t_1 \le t \le t_2$ has a single-valued solution as determined by the equation in variations (2.38) and the initial value $\delta x_1 = \delta x(t_1)$. For this reason, we shall say that $\delta x(t)$ is a <u>transposition of vector</u> $\delta x_1 = \delta x(t_1)$ given at point $x(t_1)$ along trajectory $x(t)$ and will introduce the following definition for the transposition operation which depends on parameter $\delta\mu$:

$$\delta x(t) = P_{tt_1}(\delta\mu)\delta x_1 = P_{t\tau}(\delta\mu)\delta x(\tau), \quad \tau \le t.$$

The following formulas are obvious:

$$P_{tt_1}(\gamma\delta\mu)\delta x = \gamma P_{tt_1}(\delta\mu)\delta x,\ P_{tt_1}(\delta\mu)\delta x = P_{t\tau}(\delta\mu)P_{\tau t_1}(\delta\mu)\delta x,$$
$$P_{tt_1}(\delta\mu_1 + \delta\mu_2)(\delta x_1 + \delta x_2) = P_{tt_1}(\delta\mu_1)\delta x_1 + P_{tt_1}(\delta\mu_2)\delta x_2 \qquad (2.39)$$

Let us denote by $T(x(t))$ the tangent surface to the boundary $g(x)=0$ at point $x(t)$. If $x(t)$ does not belong to the combination of neighborhoods O_{ξ_i}, which play a part in the definition of functions R (see paragraph 3, A), then it is obvious that

$$\delta x(t) = P_{tt_1}(\delta\mu)\delta x_1 \in T(x(t)).$$

In particular, it is always the case that

$$\delta x(t_2) \in T(x(t_2)).$$

For $\delta\mu = 0$ equation (2.38) becomes the homogeneous equation

$$\delta \dot{x} = \left(\frac{\partial f}{\partial x} + \Lambda\frac{\partial p}{\partial x}\right)\delta x \qquad (2.40)$$

Along with the homogeneous equation (2.40), let us examine the conjugate equation

$$\dot{\psi} = -\left(\frac{\partial f(x(t),u(t))}{\partial x} + \Lambda(t)\frac{\partial p(x(t),u(t))}{\partial x}\right)\psi \qquad (2.42)$$

If $\delta x(t), \psi(t), t_1 \leq t \leq t_2$ are arbitrary continuous solutions of equations (2.40) and (2.42), respectively, then

$$\psi(t) \cdot \delta x(t) = const, \qquad (2.43)$$

since

$$\frac{d}{dt}(\psi \delta x) = -\left[\left(\frac{\partial f}{\partial x} + \Lambda\frac{\partial p}{\partial x}\right)\psi\right] \cdot \delta x + \psi \cdot \left(\frac{\partial f}{\partial x} + \Lambda\frac{\partial p}{\partial x}\right)\delta x = 0$$

(see formulas (2.18)).

Let us denote the fundamental system of solutions to the equations (2.42) by $\varphi_0(t), \ldots, \varphi_n(t)$

and its conjugate, the system of solutions to equation (2.40), by $\psi^0(t), \ldots, \psi^n(t)$.

We have: $\psi^i(t) \cdot \varphi_j(t) = \delta^i_j$

Solution $\delta x(t)$ of the non-homogeneous equation (2.38) satisfying the initial condition

$$\delta x(t_1) = \sum_{\alpha=0}^{n} \varphi_\alpha(t_1)\delta x^\alpha(t_1),$$

is written in the form

$$\delta x(t) = P_{tt_1}(\delta\mu)\delta x(t_1) = \sum_{\alpha=0}^{n}\varphi_\alpha(t)\left(\delta x^\alpha(t_1) + \int_{t_1}^{t}\psi^\alpha(\theta)\Lambda(\theta)\frac{\partial R}{\partial\mu}\delta\mu\, d\theta\right). \qquad (2.43)$$

C) Calculation of the derivative $\dfrac{\partial R(x(t),u(t),0)}{\partial\mu}$.

We have:

$$h(x,\mu) = g\left(x + \mu\sum_{i=1}^{s} a_i(x)N_i\right) =$$

$$= g\left(x^0 + \mu\sum_{i=1}^{s} a_i(x)N_i^0, \ldots, x^n + \mu\sum_{i=1}^{s} a_i(x)N_i^n\right),$$

where $N = (N_i^0, \ldots, N^n)$. Let us introduce the definition

$$\eta^\alpha = x^\alpha + \mu\sum_{i=1}^{s} a_i(x)N_i^\alpha, \quad \eta = (\eta^0, \ldots, \eta^n).$$

Then,

$$R(x,u,\mu) = \sum_{\alpha,\beta=0}^{n} \frac{\partial g(\eta)}{\partial \eta^{\alpha}}\frac{\partial \eta^{\alpha}}{\partial x^{\beta}} f^{\beta}(x,u) = \sum_{\alpha=0}^{n} \frac{\partial g(\eta)}{\partial \eta^{\alpha}} f^{\alpha}(x,u) +$$

$$+ \mu \sum_{\alpha,\beta=0}^{n} \sum_{i=1}^{s} \frac{\partial g(\eta)}{\partial \eta^{\alpha}} \frac{\partial a_i(x)}{\partial x^{\beta}} N_i^{\alpha} f^{\beta}(x,u) = \frac{\partial g(\eta)}{\partial \eta} \cdot f(x,u) +$$

$$+ \mu \sum_{i=1}^{s} \left(\frac{\partial g(\eta)}{\partial \eta} \cdot N_i\right)\left(\frac{\partial a_i(x)}{\partial x} \cdot f(x,u)\right).$$

Consequently,

$$\frac{\partial R(x,u,0)}{\partial \mu} = \left(\sum_{\alpha,\beta=0}^{n} \frac{\partial^2 g(\eta)}{\partial \eta^{\alpha}\partial \eta^{\beta}} f^{\alpha}(x,u) \sum_{i=1} a_i(x) N_i^{\beta}\right)_{\mu=0} +$$

$$+ \sum_{i=1}^{s} \left(\frac{\partial g(x)}{\partial x} \cdot N_i\right)\left(\frac{\partial a_i(x)}{\partial x} \cdot f(x,u)\right) =$$

$$= \sum_{i=1}^{s} \sum_{\alpha,\beta=0}^{n} a_i(x) \frac{\partial^2 g(x)}{\partial x^{\alpha}\partial x^{\beta}} f^{\alpha}(x,u) N_i^{\beta} + \sum_{i=1}^{s} \left(\frac{\partial g(x)}{\partial x} \cdot N_i\right)\left(\frac{\partial a_i(x)}{\partial x} \cdot f(x,u)\right) =$$

$$= \sum_{i=1}^{s} \frac{d}{dt}\left[a_i(x(t))\left(\frac{\partial g(x(t))}{\partial x} \cdot N_i\right)\right].$$

Hence,

$$\frac{\partial R(x(t),u(t),0)}{\partial \mu} = \sum_{i=1}^{s} \frac{d}{dt}\left[a_i(x(t))\left(\frac{\partial g(x(t))}{\partial x} \cdot N_i\right)\right] \quad (2.44)$$

From formulas (2.23), (2.20) it follows that

$$\frac{\partial}{\partial \mu}\left(R_1(x,u,\mu) \oplus R_2(x,u,\mu)\right)_{\mu=0} = \frac{\partial R_1(x,u,0)}{\partial \mu} + \frac{\partial R_2(x,u,0)}{\partial \mu} \quad (2.45)$$

where the "+ sign" in the right member indicates ordinary addition.

4. Definition of varied controls and trajectories. Construction of cones K,k. The end $x(t_2)$ of trajectory $x(t)$, $t_1 \le t \le t_2$ is conditionally regular relative to point $u(t_2) = u(t_2 - 0)$. For this reason control $u(t)$ can be smoothly and continuously extended somewhat further beyond point t_2 in such a way that the continuous and regular extension of trajectory $x(t)$ does not diverge from boundary $g(x) = 0$ (see the note on the definition

of regularity, paragraph 1). We shall assume that control $u(t)$ and trajectory $x(t)$ are determined on the segment $t_1 \leq t \leq t_2 + \varepsilon\rho$ where ρ is any real number, that they satisfy the system (2.26) for $\delta\mu = 0$, and that control $u(t)$ is continuous at point t_2.

We shall construct a family Φ of permissible controls $v(t)$ defined on the segment $t_1 \leq t \leq t_2 + \varepsilon\rho$, which will be called <u>variations of the control $u(t)$, $t_1 \leq t \leq t_2 + \varepsilon\rho$</u> and will define two operations: the multiplication of the non-negative number $\gamma \geq 0$ by the element $v(t) \in \Phi$, $t_1 \leq t \leq t_2 + \varepsilon\rho$ which transposes v into the element

$$\gamma \otimes v \in \Phi, \quad t_1 \leq t \leq t_2 + \varepsilon\gamma\rho,$$

and the summation of all elements $v_i(t)$, $t_1 \leq t \leq t_2 + \varepsilon\rho_i$, $i = 1, 2$.

$$v_1 \oplus v_2 = v(t) \in \Phi, \quad t_1 \leq t \leq t_2 + \varepsilon(\rho_1 + \rho_2).$$

The symbols $\otimes$, $\oplus$ serve to emphasize the fact that the defined operations are not equivalent to the operations of multiplication by a number and addition in the vector space E^r.

The family Φ has the following properties. Let us have given the control $v(t)$, $t_1 \leq t \leq t_2 + \varepsilon\rho$, the non-negative δ parameter, a function of the form (2.21)

$$R(y, v, \varepsilon\delta\mu) = \frac{\partial h(y, \varepsilon\delta\mu)}{\partial y} \cdot f(y, v) = \frac{\partial g\left(y + \varepsilon\delta\mu \sum_{i=1}^{s} a_i(y) N_i\right)}{\partial y} \cdot f(y, v)$$

and the initial value

$$y(t_1) = x(t_1) + \varepsilon\delta x_1, \tag{2.46}$$

where

$$\delta x_1 = -\delta\mu \sum_{i=1}^{s} a_i(y(t_1)) N_i. \tag{2.47}$$

Then,

$$h(y(t_1)) = g\left(x(t_1) + \varepsilon\delta x_1 + \varepsilon\delta\mu \sum_{i=1}^{s} a_i(y(t_1)) N_i\right) - g(x(t_1)) = 0. \tag{2.48}$$

The equation (2.47) is equivalent (for small ε) to the equation

$$\delta x_1 = -\delta\mu \sum_{i=1}^{s} a_i(x(t_1)) N_i, \tag{2.49}$$

since, according to condition (see paragraph 3, A),

$$a_i(y(t_1)) = a_i(x(t_1)) = 0$$

for $\xi_i \neq x(t_1)$ and

$$a_i(y(t_1)) = a_i(x(t_1)) = 1$$

for $\xi_i = x(t_1)$

According to the listed parameters

$$v, R, \delta x_1, \rho, \delta\mu \tag{2.50}$$

and the number $\varepsilon > 0$, we obtain a single-valued <u>varied trajectory</u>

$$y(t) = y(t; v, R, \delta x_1, \rho, \delta\mu, \varepsilon), \quad t_1 \leq t \leq t_2 + \varepsilon\rho, \tag{2.51}$$

which satisfies the initial condition (2.46); the pair $v(t), y(t)$ is a solution to system (2.24) and for this reason the trajectory $y(t), t_1 \leq t \leq t_2 + \varepsilon\rho$ lies wholly in the closed region G (see paragraph 3, A).

The following expression holds at the end-point

$$y(t_2 + \varepsilon\rho) = x(t_2) + \varepsilon\delta(v, R, \delta x_1, \rho, \delta\mu) + o(\varepsilon), \tag{2.52}$$

where vector δ, whose construction is described below, is independent of and lies in $T(x(t_2))$ -- the tangential surface to the boundary $g(x)=0$ at point $x(t_2)$:

$$\delta(v, R, \delta x_1, \rho, \delta\mu) \in T(x(t_2)). \tag{2.53}$$

It further turns out that the vector δ is linear relative to its arguments in the following sense. If

$$y_i(t) = y_i(t; v_i, R_i, \delta x_i, \rho_i, \delta\mu_i, \varepsilon), \; t_1 \leq t \leq t_2 + \varepsilon\rho_i, \; i = 1,$$

are two varied trajectories, and γ_1, γ_2 are non-negative num

then the control

$$v(t) = (\gamma_1 \otimes v_1) \oplus (\gamma_2 \otimes v_2), \; t_1 \leq t \leq t_2 + \varepsilon(\gamma_1\rho_1 + \gamma_2\rho_2) = t_2$$

and the written-out value of the parameters

$$R = R_1 \oplus R_2, \; \delta x = \gamma_1\delta x_1 + \gamma_2\delta x_2, \; \rho = \gamma_1\rho_1 + \gamma_2\rho_2, \; \delta\mu = \gamma_1\delta\mu_1 + \gamma_2\delta\mu$$

have a corresponding varied trajectory

$$y(t) = y(t; v, R, \delta x, \rho, \delta\mu, \varepsilon), \quad t_1 \leq t \leq t_2 + \varepsilon\rho,$$

with end value (2.52), where

$$\delta(v, R, \delta x, \rho, \delta\mu) = \sum_{\alpha=1}^{s} \gamma_\alpha \delta(v_\alpha, R_\alpha, \delta x_\alpha, \rho_\alpha, \delta\mu_\alpha). \qquad (2.54)$$

From formulas (2.53) and (2.54) it follows that the set of all possible vectors of the form

$$x(t_2) + \delta(v, R, \delta x, \rho, \delta\mu)$$

is a convex cone K in space X^{n+1} with apex at point $x(t_2)$, lying in $T(x(t_2))$:

$$K = \{x(t_2) + \delta(v, R, \delta x, \rho, \delta\mu)\} \subset T(x(t_2)). \qquad (2.55)$$

The properties of this cone are the basis of the proofs of theorems 1 and 2 in the present study.

In the present paragraph we shall require only the subcone $k \subset K$ which is defined by the condition that the initial displacement of the varied trajectory is equal to zero: $y(t_1) = x(t_1)$, i.e., $\delta x(t_1) = 0$:

$$k = \{x(t_2) + \delta(v, R, 0, \rho, \delta\mu)\}. \qquad (2.56)$$

From formula (2.52) it follows that the corresponding ends $y(t_2 + \varepsilon\rho)$ of the varied trajectories form the same cones with a dgree of precision of up to $O(\varepsilon)$. Since $k \subset K \subset T(x(t_2))$, the cones k, K are not more than n-dimensional. For this reason, the insides of these cones will be designated by their open kernels with relation to the surface $T(x(t_2))$; their internal rays are those which contain internal points.

Since some of the structures presented below are analogous to those found in ref. 6, they will not be describe in detail.

Let τ be an internal point on the segment $t_1 \leq \tau \leq t_2$ and let there be given a set of s segments of length $\varepsilon\sigma_1, \ldots, \varepsilon\sigma_s$ where σ_i are non-negative numbers which can be numbered in such a way that the right end of the i-th segment will coicide with the left end of the (i+1)-th segment, $i = 1, \ldots, s-1$, while the right end of the s-th segment will coincide with τ. In this case we shall say that the given set of s segments is affixed to point τ.

The random variation of control $v(t)$, $t_1 \le t \le t_2 + \varepsilon\rho$ is constructed concurrently with the corresponding varied trajectory

$$y(t) = y(t;\, v, R, \delta x, \rho, \delta\mu), \quad t_1 \le t \le t_2 + \varepsilon\rho$$

of system (2.24) as soon as parameters (2.50) of the trajectory and the parameters of the control $v(t)$, which is under construction, are given; the latter will be called the <u>defining points</u>, <u>defining segments</u>, and <u>defining values</u> of control $v(t)$. Let us define these notions.

Let $\tau_1 < \ldots < \tau_k$ be random points on the interval $t_1 < t < t_2$ which differ from the discontinuities of the control u(t).

Let us call these the defining points of the control being constructed. To point τ_i, $i = 1, \ldots, k$ let there be affixed a system of s_i segments $I_{i\alpha_i}$, of lengths equal to $\varepsilon\sigma_{i\alpha_i}$, $\alpha_i = 1, \ldots, s_i$; let us call these segments the defining segments of the constructed control affixed to point τ_i. Finally, let $v_{i\alpha_i}$ be points in region Ω such that the point $x(\tau_i)$ of the unvaried trajectory is regular relative to $v_{i\alpha_i}$; we shall call $v_{i\alpha_i}$ the defining value of the constructed control which corresponds to the degining segment $I_{i\alpha_i}$.

By $\tau_{i\alpha_i}$, $\alpha_i = 1, \ldots, s_i$ let us designate the left end of segment $I_{i\alpha_i}$ and let us assume for the sake of rigor that

$$\tau_{11} \le \tau_{12} \le \ldots \le \tau_{1s_1} \le \tau_1 < \tau_{21} \le \ldots \le \tau_2 \le \ldots \le \tau_k;$$

for the sake of symmetry in the formulas let us also introduce the notation

$$\tau_{i s_{i+1}} = \tau_i, \quad i = 1, \ldots k.$$

We now proceed with the construction of functions $v(t)$, $y(t)$, $t_1 \le t \le t_2 + \varepsilon\rho$ on the basis of the given parameters of control $v(t)$ and parameters (2.50) of trajectory $y(t)$.

With the aid of the construction employed in the proof of lemma 1 (paragraph 3), we construct the solution to system (2.24) on the segment $t_1 \le t \le \tau_{11}$ on the basis of the given initial value (2.46):

$$\begin{aligned} v(t) &= u(t) + \varepsilon\delta u(t) + o(\varepsilon), \\ y(t) &= x(t) + \varepsilon\delta x(t) + o(\varepsilon). \end{aligned} \qquad (2.57)$$

Thus, $\delta x(t), t_1 \leq t \leq \tau_{11}$ is a solution of the equation in variations (2.38):

$$\delta x(t) = P_{tt_1}(\delta \mu)\delta x(t_1).$$

Point $y(\tau_{11})$ is regular relative to point ϑ_{11}, since for $\varepsilon \to 0, y(\tau_{11}) \to x(\tau_1)$. Consequently, on the segment $\tau_{11} \leq t \leq \tau_{12}$ (of length $\varepsilon \sigma_{11}$) it is possible to construct a solution $\vartheta_{11}(t), y_{11}(t)$ to system (2.24) which satisfies the following conditions:

a) $\vartheta_{11}(t)$ is a continuous differentiable function which uniformly tends toward point ϑ_{11} as $\varepsilon \to 0$;

b) $y_{11}(\tau_{11}) = y(\tau_{11})$, where the function $y(t)$, $t_1 \leq t \leq \tau_{11}$ is given by formula (2.57) (see the note following the definition of regularity, paragraph 1).

Let us extend the solution (2.57) onto the semi-interval $\tau_{11} < t \leq \tau_{12}$ with the aid of the equations

$$\vartheta(t) = \vartheta_{11}(t), \quad y(t) = y_{11}(t).$$

Repeating an analogous construction on the segments $I_{12}, \ldots, I_{1S_1}$, we shall determine the functions $\vartheta(t), y(t)$ on the segment $t_1 \leq t \leq \tau_1$. The functions $\vartheta(t), y(t)$ are again extended onto segment $\tau_1 \leq t \leq \tau_{12}$ with the aid of the construc. in lem. with the already available initial value $y(\tau_1)$ for $y(t)$ etc., all the way to point $t_2 + \varepsilon\rho$.

Near point t_2 the varied trajectory $y(t)$ is represented in the form

$$y(t) = x(t) + \varepsilon\delta x(t) + o(\varepsilon)$$

and lies on the boundary of region G, since, by agreement, the combination of closures of neighborhoods O_{ξ_i} does not contain point $x(t_2)$ and for this reason near point t_2

$$h(y(t), \varepsilon\delta\mu) = g(y(t)) = 0 \qquad (2.58)$$

Precisely in the same fashion as in ref. 6, we can prove the formula

$$y(t_2 + \varepsilon\rho; \vartheta, R, \delta x_1, \rho, \delta\mu, \varepsilon) = x(t_2) + \varepsilon\delta(\vartheta, R, \delta x_1, \rho, \delta\mu) + o(\varepsilon), \qquad (2.59)$$

where the vector δ is independent of ε and

$$\delta = \delta(\vartheta, R, \delta x_1, \rho, \delta\mu) = P_{t_2 t_1}(\delta\mu)\delta x_1 + \rho f(x(t_2), u(t_2)) +$$
$$+ \sum_{\beta=1}^{k} P_{t_2 \tau_\beta}(\delta\mu) \sum_{\alpha=1}^{S_\beta} \sigma_{\beta\alpha}\left[f(x(\tau_\beta), \vartheta_{\beta\alpha}) - f(x(\tau_\beta), u(\tau_\beta))\right]. \qquad (2.60)$$

In addition to this, it follows from (2.58) that

$$\delta(\vartheta, R, \delta x_1, \rho, \delta\mu) \in T(x(t_2)), \qquad (2.61)$$

where $T(x(t_2))$ is a tangential surface to boundary $g(x)=0$ at point $x(t_2)$.

If $\gamma \geq 0$, then the product $\gamma \otimes \vartheta$ of the number γ by control $\vartheta(t)$, $t_1 \leq t \leq t_2 + \epsilon\rho$ is constructed concurrently with the varied trajectory

$$y(t; \gamma \otimes \vartheta, R, \gamma\delta x_1, \gamma\rho, \gamma\delta\mu, \epsilon)$$

on the segment $t_1 \leq t \leq t_2 + \epsilon\gamma\rho$ by the above method, wherein the defining points of control $\gamma \otimes \vartheta$ coincide with the defining points τ_i of control ϑ, the number of defining segments $J_{i\alpha_i}$ of control $\gamma \otimes \vartheta$ affixed to τ_i remains the same, and the length of each segment is multiplied by γ:

$$\text{length } J_{i\alpha_i} = \epsilon\gamma\sigma_{i\alpha_i}, \quad \alpha_i = 1, \ldots, s_i;$$

the corresponding defining values are unchanged -- the segment $J_{i\alpha}$ has a corresponding value $\vartheta_{i\alpha_i}$. Equation (2.47) continues to hold for new parameters:

$$\gamma\delta x_1 = -\gamma\delta\mu \sum_{i=1}^{s} a_i(x(t_1) + \epsilon\gamma\delta x_1)N_i = -\gamma\delta\mu \sum_{i=1}^{s} a_i(x(t_1))N_i.$$

It follows directly from formula (2.60) that:

$$\delta(\gamma \otimes \vartheta, R, \gamma\delta x_1, \gamma\rho, \gamma\delta\mu) = \gamma\delta(\vartheta, R, \delta x_1, \rho, \delta\mu). \qquad (2.62)$$

The sum $\vartheta = \vartheta_1 \oplus \vartheta_2$ of two varied controls

$$\vartheta_i(t), \quad t_1 \leq t \leq t_2 + \epsilon\rho_i, \quad i = 1, 2,$$

which have corresponding varied trajectories

$$y_i(t; \vartheta_i, R_i, \delta x_i, \rho_i, \delta\mu_i), \quad t_1 \leq t \leq t_2 + \epsilon\rho_i, \quad i = 1, 2,$$

is constructed on the segment $t_1 \leq t \leq t_2 + \epsilon(\rho_1 + \rho_2)$ concurrently with the corresponding varied trajectory

$$y(t; \vartheta_1 \oplus \vartheta_2, R_1 \oplus R_2, \delta x_1 + \delta x_2, \rho_1 + \rho_2, \delta\mu_1 + \delta\mu_2, \epsilon)$$

by the standard method described above, wherein the defining points of control $\vartheta_1 \oplus \vartheta_2$ are obtained through the combination of all of the defining points of the summed controls; to each defining point τ of control $\vartheta_1 \oplus \vartheta_2$ thus obtained are affixed (without changing their length) all of the defining segments affixed to τ in the summed controls;

finally, to each of the defining segments of the sum control taken from a summed control ϑ_i is ascribed the same defining significance that it has with reference to the summed ϑ_i.

Equation (2.47) is satisfied for the new parameters (see the definition of functions $h_1 \oplus h_2, R_1 \oplus R_2$, and the fromulas (2.20) and (2.23)).

Taking into account (2.60) and (2.39), we easily derive the formula

$$\delta(\vartheta_1 \oplus \vartheta_2, R_1 \oplus R_2, \delta x_1 + \delta x_2, \rho_1 + \rho_2, \delta\mu_1 + \delta\mu_2) =$$

$$= \sum_{\alpha=1}^{2} \delta(\vartheta_\alpha, R_\alpha, \delta x_\alpha, \rho_\alpha, \delta\mu_\alpha),$$

which, along with formula (2.62) yields (2.54). Thus, the cones K, k have been constructed.

The inclusive relationship

$$K \subset T(x(t_2))$$

follows from (2.61).

Let us note that the controls $\gamma \otimes \vartheta, \vartheta_1 \oplus \vartheta_2$ determined on the basis of the aforementioned conditions are give with precision of an order wherein the defining segments are affixed to the defining point. As may easily be seen, however, this order of accuracy does not affect the magnitude of vector (2.60), and consequently, the cones K, k are single-valued.

5. Proof of theorem 1. Let us suppose that control and the regular trajectory are optimal.

Let us denote by S the ray which emerges from point and is directed along the negative axis ; it is obvious that .

Lemma 2. Ray S is not an internal ray of cone K.

Proof. Let us assume that S is an internal ray for K. It is obvious, then, that it is possible to select n varied trajectories

$$y_i(t), \quad t_1 \le t \le t_2 + \epsilon\rho_i, \quad i = 1, \dots, n,$$

having null initial deviations $y_i(t_1) = x(t_1)$ in such a way that the corresponding vectors δ_i, $i = 1, \dots, n$ given by formula (2.52) will generate the n-dimensional cone with apex at point $x(t_2)$, enclosing ray S within it -- the cone

$$\{x(t_2)+\gamma_1\delta_1+\dots+\gamma_n\delta_n\},\qquad (2.63)$$

where $\gamma_i \geq 0$.

From the constructions of paragraph 4 follows the existence of an (n+1)-parameter family of varied trajectories

$$y(t;\gamma_1,\dots,\gamma_n,\varepsilon),\quad t_1 \leq t \leq t_2+\varepsilon\rho(\gamma_1,\dots,\gamma_n),$$

continuously dependent on parameters $\varepsilon,\gamma_1,\dots,\gamma_n$ and defined for

$$0 \leq \varepsilon \leq \varepsilon_1,\quad 0 \leq \gamma_i \leq \gamma^*,\quad i=1,\dots,n,$$

such that the trajectories of the family satisfy the initial conditions

$$y(t_1;\gamma_1,\dots,\gamma_n,\varepsilon)=x(t_1) \text{ for any } \varepsilon,\gamma_i$$

and the final conditions

$$y(t_2+\varepsilon\rho;\gamma_1,\dots,\gamma_n,\varepsilon)=x(t_2)+\varepsilon\sum_{\alpha=1}^{n}\gamma_\alpha\delta_\alpha+o(\varepsilon).\qquad (2.64)$$

By construction, the parallelepiped

$$\Pi=\{x(t_2)+\sum_{\alpha=1}^{n}\gamma_\alpha\delta_\alpha\},\quad 0\leq\gamma_i\leq\gamma^*,\quad i=1,\dots,n,$$

encloses within it a portion of ray S.

Let Θ be some projection of the neighborhood of point $x(t_2)$ (relative to X^{n+1}) on plane $T(x(t_2))$. We can assume that this neighborhood contains all points (2.64), since the numbers ε_1,γ^* can be arbitrarily small.

Obviously,

$$\Theta y(t_2+\varepsilon\rho;\gamma_1,\dots,\gamma_n,\varepsilon)=\Theta\Big(x(t_2)+\varepsilon\sum_{\alpha=1}^{n}\gamma_\alpha\delta_\alpha+o(\varepsilon)\Big)=$$

$$=\Theta\Big(x(t_2)+\varepsilon\sum_{\alpha=1}^{n}\gamma_\alpha\delta_\alpha\Big)+o(\varepsilon)=x(t_2)+\varepsilon\sum_{\alpha=1}^{n}\gamma_\alpha\delta_\alpha+o(\varepsilon),$$

since for any point $y\in T(x(t_2))$

$$\Theta y=y.$$

Let us define the continuous mapping $\Gamma(\gamma,\varepsilon)=\Gamma(\gamma_1,\dots,\gamma_n,\varepsilon)$ by means of the equations:

$$\Gamma(\gamma_1, \ldots, \gamma_n, \varepsilon) = x(t_2) + \frac{\theta y(t_2+\varepsilon\rho; \gamma, \varepsilon) - x(t_2)}{\varepsilon} =$$

$$= x(t_2) + \sum_{\alpha=1}^{n} \gamma_\alpha \delta_\alpha + \frac{o(\varepsilon)}{\varepsilon} \quad \text{for } \varepsilon > 0$$

$$\Gamma(\gamma_1, \ldots, \gamma_n, 0) = x(t_2) + \sum_{\alpha=1}^{n} \gamma_\alpha \delta_\alpha .$$

Let $(-\eta, 0, \ldots, 0), \eta > 0$ be a random internal point of ray S enclosed within Π ; it is enveloped in the form of a cube $0 \le \gamma_i \le \gamma^*, i=1,\ldots,n$, for the mapping Γ_i, with index 1. Consequently, the equation relative to $\gamma_i, 0 \le \gamma_i \le \gamma^*$, $i=1,\ldots,n$,

$$\Gamma(\gamma_1, \ldots, \gamma_n, \varepsilon) - x(t_2) = (-\eta, 0, \ldots, 0)$$

for sufficiently small ε has at least one solution. For $\varepsilon > 0$ the latter equation is equivalent to the equation

$$\theta y(t_2+\varepsilon\rho; \gamma, \varepsilon) = x(t_2) + \varepsilon(-\eta, 0, \ldots, 0).$$

A point in the right member of the equation simultaneously belongs to boundary $g(x)=0$ and the tangential surface $T(x(t_2))$; hence, the following equation, which contradicts the assumption regarding the optimality of trajectory $x(t_2)$ holds:

$$\theta y(t_2+\varepsilon\rho; \gamma, \varepsilon) = y(t_2+\varepsilon\rho; \gamma, \varepsilon) = x(t_2) + \varepsilon(-\eta, 0, \ldots, 0),$$

Lemma 3. There exists a continuous solution

$$\psi(t) = (\psi_0(t), \ldots, \psi_n(t)), \quad t_1 \le t \le t_2,$$

to the equation

$$\dot{\psi} = -\left(\frac{\partial f(x(t), u(t))}{\partial x} + \Lambda(t)\frac{\partial p(x(t), u(t))}{\partial x}\right)\psi \quad (2.65)$$

such that in each point wherein control u(t) is continuous the maximum condition

$$H(\psi(t), x(t), u(t)) = m(\psi(t), x(t)), \quad (2.66)$$

is satisfied; furthermore,

$$m(\psi(t_2), x(t_2)) = 0 \quad (2.67)$$

In addition to this, the following conditions are fulfilled:

a) $\psi_0(t) = \text{const} \leq 0$,

b) vector $\psi(t_1)$ is not collinear to vector $\operatorname{grad} g(x(t_1))$,

c) at the points of differentiability of the piecewise-smooth scalar function, $\lambda(t) = -\psi(t) \cdot \Lambda(t), t_1 \leq t \leq t_2$, the vector

$$\frac{d\lambda(t)}{dt} \operatorname{grad} g(x(t))$$

is directed into region G or goes to zero.

Proof. On the basis of lemma 2, through apex $x(t_1)$ of cone $k \subset T(x(t_2))$ it is possible to pass an (n-1)-dimensional reference surface to k lying in $T(x(t_2))$ and separating cone k from ray S. The vector which is orthogonal to this surface, lies in $T(x(t_2))$, and is directed toward ray S will be denoted by $\chi = (\chi_0, \dots, \chi_n)$. We have:

$$\chi\ (-1, 0, \dots, 0) = -\chi_0 \geq 0,$$

i.e., $\chi_0 \leq 0$. Since χ lies in $T(x(t_2))$, the vectors

$$\chi\ , \quad \operatorname{grad} g(x(t_2)) \tag{2.68}$$

are independent.

Vector δ which according to formula (2.52) corresponds to the random varied trajectory $y(t)$ with the initial condition $y(t_1) = x(t_1)$, satisfies the inequality

$$\chi \cdot \delta \leq 0. \tag{2.69}$$

The required function $\psi(t), t_1 \leq t \leq t_2$ is defined as the solution to (2.65) which satisfies the boundary condition

$$\psi(t_2) = \chi. \tag{2.70}$$

Indeed, let us have that at a certain point of discontinuity τ of control u(t)

$$H(\psi(\tau), x(\tau), u(\tau)) < m(\psi(\tau), x(\tau)),$$

in other words, there exists a point $v_1 \in \Omega$ relative to which the point $x(\tau)$ is regular and

$$H(\psi(\tau), x(\tau), v_1) > H(\psi(\tau), x(\tau), u(\tau)). \tag{2.71}$$

Let us construct a solution $v(t), y(t), t_1 \leq t \leq t_2$ to system (2.24), setting as parameters for the control v a single defining segme of length ε affixed to a single defining point τ , and a

$$\Gamma(r_1,\ldots,r_n,\varepsilon)=x(t_2)+\frac{\theta y(t_2+\varepsilon\rho;r,\varepsilon)-x(t_2)}{\varepsilon}=$$

$$=x(t_2)+\sum_{\alpha=1}^{n}r_\alpha\delta_\alpha+\frac{o(\varepsilon)}{\varepsilon}\quad\text{for } \varepsilon>0$$

$$\Gamma(r_1,\ldots,r_n,0)=x(t_2)+\sum_{\alpha=1}^{n}r_\alpha\delta_\alpha.$$

Let $(-\eta,0,\ldots,0),\ \eta>0$ be a random internal point of ray S enclosed within Π; it is enveloped in the form of a cube $0\leq r_i\leq r^*, i=1,\ldots,n$, for the mapping Γr_i with index 1. Consequently, the equation relative to r_i, $0<r_i<r_i^*$, $i=1,\ldots,n$,

$$\Gamma(r_1,\ldots,r_n,\varepsilon)-x(t_2)=(-\eta,0,\ldots,0)$$

for sufficiently small ε has at least one solution. For $\varepsilon>0$ the latter equation is equivalent to the equation

$$\theta y(t_2+\varepsilon\rho;r,\varepsilon)=x(t_2)+\varepsilon(-\eta,0,\ldots,0).$$

A point in the right member of the equation simultaneously belongs to boundary $g(x)=0$ and the tangential surface $T(x(t_2))$; hence, the following equation, which contradicts the assumption regarding the optimality of trajectory $x(t_2)$ holds:

$$\theta y(t_2+\varepsilon\rho;r,\varepsilon)=y(t_2+\varepsilon\rho;r,\varepsilon)=x(t_2)+\varepsilon(-\eta,0,\ldots,0),$$

Lemma 3. There exists a continuous solution

$$\psi(t)=(\psi_0(t),\ldots,\psi_n(t)),\quad t_1\leq t\leq t_2,$$

to the equation

$$\dot\psi=-\left(\frac{\partial f(x(t),u(t))}{\partial x}+\Lambda(t)\frac{\partial p(x(t),u(t))}{\partial x}\right)\psi \tag{2.65}$$

such that in each point wherein control u(t) is continuous the maximum condition

$$H(\psi(t),x(t),u(t))=m(\psi(t),x(t)), \tag{2.66}$$

is satisfied; furthermore,

$$m(\psi(t_2),x(t_2))=0 \tag{2.67}$$

In addition to this, the following conditions are fulfilled:

a) $\psi_0(t) = \text{const} \leq 0$,

b) vector $\psi(t_1)$ is not collinear to vector $\text{grad}\, g(x(t_1))$,

c) at the points of differentiability of the piecewise-smooth scalar function, $\lambda(t) = -\psi(t)\cdot\Lambda(t), t_1 \leq t \leq t_2$, the vector

$$\frac{d\lambda(t)}{dt}\,\text{grad}\, g(x(t))$$

is directed into region G or goes to zero.

Proof. On the basis of lemma 2, through apex $x(t_1)$ of cone $k \subset T(x(t_2))$ it is possible to pass an (n-1)-dimensional reference surface to k lying in $T(x(t_2))$ and separating cone k from ray S. The vector which is orthogonal to this surface, lies in $T(x(t_2))$, and is directed toward ray S will be denoted by $\chi = (\chi_0, \ldots, \chi_n)$. We have:

$$\chi\,(-1, 0, \ldots, 0) = -\chi_0 \geq 0,$$

i.e., $\chi_0 \leq 0$. Since χ lies in $T(x(t_2))$, the vectors

$$\chi\,, \quad \text{grad}\, g(x(t_2)) \tag{2.68}$$

are independent.

Vector δ which according to formula (2.52) corresponds to the random varied trajectory $y(t)$ with the initial condition $y(t_1) = x(t_1)$, satisfies the inequality

$$\chi\cdot\delta \leq 0. \tag{2.69}$$

The required function $\psi(t), t_1 \leq t \leq t_2$ is defined as the solution to (2.65) which satisfies the boundary condition

$$\psi(t_2) = \chi. \tag{2.70}$$

Indeed, let us have that at a certain point of discontinuity τ of control u(t)

$$H(\psi(\tau), x(\tau), u(\tau)) < m(\psi(\tau), x(\tau)),$$

in other words, there exists a point $v_1 \in \Omega$ relative to which the point $x(\tau)$ is regular and

$$H(\psi(\tau), x(\tau), v_1) > H(\psi(\tau), x(\tau), u(\tau)). \tag{2.71}$$

Let us construct a solution $v(t), y(t), t_1 \leq t \leq t_2$ to system (2.24), setting as parameters for the control v a single defining segme of length ε affixed to a single defining point τ, and a

defining value v_1; the parameters for trajectory $y(t)$ are given by the system

$$R = p(x, u), \quad \delta x = 0 \; , \quad \rho = \delta_\mu = 0 .$$

From (2.60) it follows that the vector for this trajectory has the form:

$$\delta = P_{t_2 \tau}(0) \, [f(x(\tau), v_1) - f(x(\tau), u(\tau))] .$$

Consequently, from (2.42) we obtain:

$$\begin{aligned} \chi \cdot \delta = \psi(t_2) \cdot P_{t_2 \tau}(0) [f(x(\tau), v_1) - f(x(\tau), u(\tau))] = \\ = \psi(\tau) \cdot [f(x(\tau), v_1) - f(x(\tau), u(\tau))] \leq 0 , \end{aligned}$$

which contradicts inequality (2.71).

To prove formula (2.67) we construct the varied trajectory

$$y(t), \quad t_1 \leq t \leq t_2 + \epsilon \rho ,$$

with parameters $R = p(x, u), \delta x = 0, \delta_\mu = 0 \; ; \; \rho$, which corresponds to the varied control

$$v(t) \equiv u(t) \; , \quad t_1 \leq t \leq t_2 + \epsilon \rho .$$

The vector δ for this trajectory has the form

$$\delta = \rho f(x(t_2), u(t_2)) .$$

(2.67) follows from equations (2.66), (2.70), and the inequality

$$\chi \cdot \rho f(x(t_2), u(t_2)) \leq 0$$

.

Condition a) of lemma 3 follows from inequality $\chi_0 \leq 0$ and the independence of the right member of (2.65) from χ^0 .

Condition b) follows from the independence of vectors (2.68).

Let us prove condition c). Let $v(t), y(t), t_1 \leq t \leq t_2$ be the solution of system (2.24) obtained by means of the construction of paragraph 3 (lemma 1) and satisfying the initial value

$$y(t_1) = x(t_1).$$

Then $v(t)$ is a varied control (in the sense of paragraph 4) without defining points, while $y(t) = y(t; v, R, 0, 0, \delta_\mu, \epsilon$

is the corresponding varied trajectory corresponding to the parameters $R, \delta x_0 = 0, \rho = 0, \delta\mu$. From formulas (2.60) and (2.43) it follows that

$$\delta(\sigma, R, 0, 0, \delta\mu) = \delta = P_{t_2 t_1}(\delta\mu) 0 = \delta\mu \sum_{\alpha=0}^{n} p_\alpha(t_2) \int_{t_1}^{t_2} \psi^\alpha(t) \cdot \Lambda(t) \frac{\partial R}{\partial \mu} dt.$$

Formulas (2.70) and (2.69) yield:

$$\varkappa \cdot \delta = \delta\mu \sum_{\alpha=0}^{n} \psi(t_2) \cdot p_\alpha(t_2) \int_{t_1}^{t_2} \psi^\alpha(t) \cdot \Lambda(t) \frac{\partial R}{\partial \mu} dt =$$

$$= \delta\mu \int_{t_1}^{t_2} \psi(t) \cdot \Lambda(t) \frac{\partial R}{\partial \mu} \cdot dt = -\delta\mu \int_{t_1}^{t_2} \lambda(t) \frac{\partial R}{\partial \mu} dt \leq 0.$$

Substituting in this equation the expression for $\frac{\partial R}{\partial \mu}$ from (2.44), we obtain $(\delta\mu \geq 0)$:

$$\int_{t_1}^{t_2} \lambda(t) \frac{d}{dt} \sum_{i=1}^{s} \left[a_i \varkappa(t) \left(\frac{\partial g(\varkappa(t))}{\partial \varkappa} \cdot N_i \right) \right] \geq 0.$$

Integrating by parts and taking into account the equations

$$a_i(\varkappa(t_1)) = a_i(\varkappa(t_2)) = 0$$

(for in the case where $\delta x_0 = 0$ not one of the neighborhoods O_{ζ_i} which play a part in the definition of function R contains points $\varkappa(t_1) \varkappa(t_2)$), we obtain:

$$\int_{t_1}^{t_2} \frac{d\lambda}{dt} \sum_{i=1}^{s} a_i \varkappa(t) \left(\frac{\partial g(\varkappa(t))}{\partial \varkappa} \cdot N_i \right) dt \leq 0.$$

Since the points ζ_i can be selected on trajectory $\varkappa(t)$ at random, provided that these points do not coincide with the ends of the trajectory, and since the neighborhoods O_{ζ_i} are arbitrarily small and the functions $a_i \varkappa(t)$ non-negative, from the last inequality there follows the following inequality

$$\frac{d\lambda(t)}{dt} \leq 0$$

which in the present case is equivalent to condition c), since vector $\operatorname{grad}(\varkappa(t))$ serves as the external normal to region G.

To complete the proof of theorem 1 it is necessary, first of all to establish that the function $\lambda(t) = -\psi(t) \cdot \lambda(t)$ satisfies relation (2.13):

$$\frac{\partial H(\psi(t), \varkappa(t), u(t))}{\partial u} = \lambda(t) \frac{\partial p(\varkappa(t), u(t))}{\partial u} + \sum_{\alpha=1}^{s} \nu_\alpha(t) \frac{\partial q_\alpha(u(t))}{\partial u}, \quad (2.72)$$

and secondly to prove the equation

$$H(\psi(t), \chi(t), u(t-0)) = m(\psi(t), \chi(t))$$

at the control rupture points; finally, it is necessary to establish the constancy of function $H(t) = H(\psi(t), \chi(t), u(t))$ on the segment $t_1 \le t \le t_2$:

$$H(t) = H(\psi(t), \chi(t), u(t)) = m(\psi(t), \chi(t)) = \text{const}, \quad t_1 \le t \le t_2,$$

where, according to the condition assumed,

$$H(t) = H(t-0)$$

at the control rupture points.

To prove equation (2.72), let us note that the coordinates of vector $\Lambda(t) = (\lambda^0(t), \ldots, \lambda^n(t))$ satisfy equations (2.35):

$$\frac{\partial f^j(\chi(t), u(t))}{\partial u^\alpha} + \lambda^j(t)\frac{\partial p(\chi(t), u(t))}{\partial u^\alpha} + \sum_{\beta=1}^{s} \ell_\beta^j(t)\frac{\partial q_\beta(u(t))}{\partial u^\alpha} = 0$$

$$(j = 0, \ldots, n, \quad \alpha = 1, \ldots, s+1).$$

Summing these equations with $\psi_j(t)$, $j = 0, \ldots, n$, over index j, we obtain $s+1$ equations

$$\frac{\partial H(\psi(t), \chi(t), u(t))}{\partial u^\alpha} = \lambda(t)\frac{\partial p(\chi(t), u(t))}{\partial u^\alpha} + \sum_{\beta=1}^{s} \nu_\beta(t)\frac{\partial q_\beta(u(t))}{\partial u^\alpha}$$

from which results a single-valued function $\lambda(t)$ (along with functions $\nu_\beta(t)$, $\beta = 1, \ldots, s$).

On the other hand,

$$H(\psi(t), \chi(t), u(t)) = m(\psi(t), \chi(t)),$$

and, according to the rule for Lagrange multipliers,

$$\frac{\partial H(\psi(t), \chi(t), u(t))}{\partial u} = \lambda^*(t)\frac{\partial p(\chi(t), u(t))}{\partial u} + \sum_{\alpha=1}^{s} \nu_\alpha^* \frac{\partial q_\alpha(u(t))}{\partial u},$$

consequently,

$$\lambda(t) = \lambda^*(t), \; \nu_\alpha(t) = \nu_\alpha^*(t), \quad \alpha = 1, \ldots, s.$$

Let us now prove the equation

$$H(\psi(t), \chi(t), u(t-0)) = m(\psi(t), \chi(t))$$

at points of rupture of control $u(t)$.

The function $H(\psi(t), \chi(t), u)$ is continuous with respect to t for fixed u. Let τ be a rupture point in control $u(t)$. By agreement, $u(\tau) = u(\tau - 0)$ and the point $\chi(\tau)$ is regular relative to points $u(\tau \pm 0)$. So, if

$$H(\psi(\tau), \chi(\tau), u(\tau)) < m(\psi(\tau), \chi(\tau)),$$

there will be found a point $u_1 \in \omega(x(\tau))$ such that

$$H(\psi(\tau), x(\tau), u(\tau)) < H(\psi(\tau), x(\tau), u_1).$$

Function $u(t)$ is continuous at point τ from the left; consequently, for any point $t < \tau$ sufficiently close to τ:

$$H(\psi(t), x(t), u(t)) < H(\psi(t), x(t), u_1).$$

Since point t is close to τ and $u_1 \in \omega(x(t))$,

$$u_1 \in \omega(x(t))$$

and for this reason the last inequality yields:

$$H(\psi(t), x(t), u(t)) < H(\psi(t), x(t), u_1) \leq m(\psi(t), x(t)),$$

in contradiction to lemma 3.

To prove the constancy of function $H(t) = m(\psi(t), x(t))$ let us first prove that it is continuous. To do this, it is sufficient to show that at every control rupture point τ

$$H(\psi(\tau), x(\tau), u(\tau - 0)) = H(\psi(\tau), x(\tau), u(\tau + 0)).$$

Let $t > \tau$ be a point close to τ; then

$$H(\psi(t), x(t), u(\tau - 0)) \leq H(\psi(t), x(t), u(t)) = m(\psi(t), x(t)),$$

since the point $x(t)$ is regular relative to point $u(\tau - 0)$. Consequently,

$$H(\psi(\tau), x(\tau), u(\tau - 0)) \leq H(\psi(\tau), x(\tau), u(\tau + 0)).$$

The converse inequality is proved in an analogous manner.

Let us show that function $H(t)$ is constant over each interval in which functions $\psi(t), x(t), u(t)$ are simultaneously differentiable. Indeed,

$$\frac{d}{dt} H(t) = \dot{\psi} \cdot \frac{\partial H}{\partial \psi} + \frac{\partial H}{\partial x} \cdot \dot{x} + \frac{\partial H}{\partial u} \cdot \dot{u} = \left(-\frac{\partial f}{\partial x} \psi + \lambda \frac{\partial p}{\partial x}\right) \cdot f +$$

$$+ \left(\frac{\partial f}{\partial x} \psi\right) \cdot f + \lambda \frac{dp}{du} + \sum_{\alpha=1}^{s} \nu_\alpha \frac{\partial q_\alpha}{\partial u}) \cdot \dot{u} = \lambda \frac{\partial p}{\partial x} \cdot f + \lambda \frac{\partial p}{\partial u} \cdot u +$$

$$+ \sum_{\alpha=1}^{s} \nu_\alpha \frac{\partial q_\alpha}{\partial u} \cdot \dot{u} = \lambda \frac{d}{dt} p(x(t), u(t)) + \sum_{\alpha=1}^{s} \nu_\alpha \frac{d q_\alpha(u(t))}{dt} = 0,$$

q.e.d.

6. Generalizations. In this section we will describe several obvious generalizations of theorem 1. We will limit ourselves to the formulation of results, since the proofs differ little from the proof of theorem 1.

In proving theorem 1 the decisive role was played by the relationship between x and u given by the equation

$$p(x, u) = 0$$

The form of the function

$$p(x, u) = \frac{\partial g(x)}{\partial x} \cdot f(x, u)$$

was used only in proving conditions b) and c) of theorem 1. A cursory analysis of the proof convinces us of the correctness of the following theorem 1,a.

Let there be given m continuously differentiable functions $p_i(x, u)$, $i = 1, \dots, m$ independent of coordinate x^0. Point x, regular relative to point $u_0 \in \Omega$, which satisfies the system

$$p_1(x, u_0) = \dots = p_m(x, u_0) = 0,$$

is determined as before, except that in the present case instead of requiring that vectors (2.9) be independent, we require the independence of vectors

$$\frac{\partial p_1(x, u_0)}{\partial u}, \dots, \frac{\partial p_m(x, u_0)}{\partial u}, \frac{\partial q_1(u_0)}{\partial u}, \dots, \frac{\partial q_s(u_0)}{\partial u}.$$

Theorem 1,a. Let $u(t)$, $t_0 \leq t \leq t_1$ be an optimal control and $x(t)$ the corresponding regular optimal trajectory of equation (1.5) satisfying the following system of equations on the segment $t_0 \leq t \leq t_1$:

$$p_1(x(t), u(t)) = \dots = p_m(x(t), u(t)) = 0 \qquad (2.73)$$

Then there exists a non-zero continuous covariant vector function $\psi(t) = (\psi_0(t), \dots, \psi_n(t))$, $t_0 \leq t \leq t_1$ such that on the segment $t_0 \leq t \leq t_1$ the following system of equations is satisfied:

$$\dot{x} = f(x, u) = \frac{\partial H(\psi, x, u)}{\partial \psi}, \quad \dot{\psi} = -\frac{\partial H(\psi, x, u)}{\partial x} + \sum_{s=1}^{m} \lambda_s \frac{\partial p_s(x, u)}{\partial x}$$

and the maximum condition

$$H(\psi(t), x(t), u(t)) = M(\psi(t), x(t))$$

is fulfilled; furthermore

$$M(\psi(t), x(t)) = 0 \quad ;$$

the piecewise-smooth functions $\lambda_s(t)$, $s = 1, \dots, m$, $t_0 \leq t \leq t_1$

are determined from the maximum condition as Lagrange multipliers in the formula

$$\frac{\partial H(\psi, x, u)}{\partial u} = \sum_{\alpha=1}^{r} \lambda_\alpha \frac{\partial p_\alpha(x,u)}{\partial u} + \sum_{\alpha=1}^{s} \nu_\alpha \frac{\partial \varphi_\alpha(u)}{\partial u},$$

while the coordinate

$$\psi_0(t) = \text{const} \le 0.$$

The solution of the optimal which in place of equations (2.73) incorporates the inequalities

$$p_1(x, u) \le 0, \dots, p_m(x, u) \le 0. \qquad (2.74)$$

is easily reduced to the theorem formulated above.

Indeed, introducing m additional scalar control parameters v_i, $i = 1, \dots, m$, satisfying the inequalities $v_i \ge 0$, and considering in place of inequalities (2.74) the equations

$$p_1(x, u) + v_1 = \dots = p_m(x, u) + v_m = 0,$$

we arrive at the conditions of theorem 1,a.

Finally, we note that theorem 1 is directly generalized in the case where region G is given near the boundary by several, let us say two, inequalities

$$g_1(x) \le 0, \quad g_2(x) \le 0,$$

while the regular optimal trajectory lies on the (n-2)-dimensional "ridge"

$$g_1(x) = g_2(x) = 0;$$

it is of course assumed at this point that the hypersurface

$$g_1(x) = 0, \quad g_2(x) = 0$$

is generally oriented along the trajectory, i.e., that the vectors $\frac{\partial g_1(x)}{\partial x}$, $\frac{\partial g_2(x)}{\partial x}$ are independent.

3. Jump Conditions

The optimal trajectory which lies in the closed region G can lie partially in the open kernel of region G and partially on the boundary. In order to make a single-valued trace of such a trajectory, it is insufficient merely to apply the Maximum Principle and theorem 1. Actually, the Maximum Principle gives a complete system of necessary conditions which

must be satisfied by any portion of the optimal trajectory lying wholly in the open kernel of region G, while theorem 1 gives the necessary conditions which are satisfied by the portions lying wholly on the boundary of region G. We still lack a conjugation condition satisfied by any pair of adjacent portions of the optimal trajectory, one of which lies in the open kernel of region G and the other on its boundary. This condition will be called the jump condition for the covariant vector function $\psi(t)$ which at the moment of passage from one portion to the other can exhibit a discontinuity (see the formulation of theorem 2).

For purposes of easy reference, paragraph 1 below contains a formulation of the Maximum Principle and the proof of a supplementary lemma. Paragraph 2 contains basic definitions, and paragraph 3 -- a basic formulation and proof of theorem 2, the jump condition.

1. The Maximum Principle. Let $x(t), t_1 \leq t \leq t_2$ be the optimal trajectory of equation (1.5) lying wholly in the open kernel of region G; $u(t), t_1 \leq t \leq t_2$ is the corresponding optimal control. Then there will be found a continuous non-zero function $\psi(t) = (\psi_0(t), \ldots, \psi_n(t)), t_1 \leq t \leq t_2$ such that the inequality

$$\psi_0(t) = \text{const} \leq 0$$

and the system of equations

$$\dot{x} = f(x, u) = \frac{\partial H(\psi, x, u)}{\partial \psi}, \tag{3.1}$$

$$\dot{\psi} = -\frac{\partial f(x, u)}{\partial x}\psi = -\frac{\partial H(\psi, x, u)}{\partial x}, \tag{3.2}$$

$$H(\psi(t), x(t), u(t)) = M(\psi(t), x(t)) = 0 \tag{3.3}$$

where the value of $M(\psi, x)$ is given by formula (2.2), are satisfied on the segment $t_1 \leq t \leq t_2$.

The Maximum Principle holds in the case when one or both of the ends $x(t_1), x(t_2)$ of the trajectory lie on boundary $g(x) = 0$. For example, let only the initial point $x(t_1)$ of the trajectory lie on the boundary. Then for $t_1 + \theta \leq t \leq t_2, \theta > 0$ a portion of trajectory $x(t)$ lies in the open kernel of region G, and consequently there exists a function $\psi_\theta(t)$, $t_1 + \theta \leq t \leq t_2$ which satisfies the requirements of the Maximum Principle on the segment $t_1 + \theta \leq t \leq t_2$. For the family of functions $\psi_\theta(t)$ as $\theta \to 0$ there exists a limiting function $\psi(t), t_1 \leq t \leq t_2$ which will be the one that is sought.

A detailed proof of the Maximum Principle is found in ref. 6. The basic role in this proof is played by the **transposition construction along** the optimal trajectory $x(t)$, $t_1 \le t \le t_2$ with the aid of the equation in variations

$$\delta \dot{x} = \frac{\partial f(x(t), u(t))}{\partial x} \delta x \tag{3.4}$$

for equation (3.1) and the construction of varied controls v*(t) and varied trajectories y*(t) which differ little from optimal control u(t) and optimal trajectory $x(t)$.

If the fundamental system of solutions to equation (3.4) is denoted as

$$\varphi_0^*(t), \dots, \varphi_n^*(t), \quad t_1 \le t \le t_2,$$

the vector

$$\delta^* x(t_1) = \sum_{\alpha=0}^{n} \varphi_\alpha^*(t_1) \delta^* x^\alpha(t_1),$$

given at point $x(t_1)$, after transposition along the trajectory to point $x(t)$ with the aid of equation (3.4), passes into the vector

$$P^*_{tt_1} \delta^* x(t_1) = \sum_{\alpha=0}^{n} \varphi_\alpha^*(t) \delta^* x^\alpha(t_1).$$

The constructions of section 2 represent a complication of the analogous constructions in the case under consideration, since in section 2, in addition to the satisfaction of equation (1.5) we also required the fulfillment of the following equality:

$$R(y; \tau, \varepsilon, u) = 0.$$

If we discard this last equation, system (2.24) becomes the single equation (3.1), while the equation in variations (2.38) for the system (2.24) is replaced by the equation in variations (3.4).

In accordance with this, in place of cones (2.55), (2.56) we obtain cones (3.5), (3.6) with apexes at point $x(t_2)$:

$$K^* = \{x(t_2) + \delta^*(v^*, \delta^* x, \rho^*)\}, \tag{3.5}$$

$$k^* = \{x(t_2) + \delta^*(v^*, 0, \rho^*)\}, \tag{3.6}$$

where the vector δ^* is defined by the formula (3.7) which is analogous to formula (2.60):

$$\delta^* = \sum_{i=1}^{k} \rho^*_{\tau_i \tau_i}\left[\sum_{\alpha_i=1}^{s}\sigma_{i\alpha_i}\left(f(x(\tau_i), v^*_{i\alpha_i}) - f(x(\tau_i); u(\tau_i))\right)\right] +$$

$$+ \sum_{\alpha=0}^{n} \varphi^*_\alpha (t_2)\,\delta^* x^\alpha(t_1) + \rho^* f(x(t_2), u(t_2)). \qquad (3.7)$$

The proof of the Maximum Principle involves only cone K^* corresponding to the null initial deviations. Cone K^* will be needed in paragraph 3.

As proof of this point we shall one simple lemma which will be needed in paragraph 3.

Lemma. Let $x(t), t_1 \leq t \leq t_2$ be the trajectory of equation (1.5) lying in the closed region G and corresponding to some permissible control, and let $x(t_1)$ be the only point in the trajectory lying on the boundary $g(x)=0$ of region G. If the vector $\delta x(t_1)$ is not tangent to boundary $g(x)=0$ at the point $x(t_1)$ and is directed into region G, then the varied trajectory $y(t), t_1 \leq t \leq t_2$ with the initial value $y(t_1) = x(t_1) + \varepsilon\delta x(t_1)$ lies whooly in the open kernel of region G.

Proof. We have:

$$y(t) = x(t) + \varepsilon \delta x(t) + o(\varepsilon), \quad \frac{\partial g(x(t_1))}{\partial x} \cdot \delta x(t_1) = a < 0.$$

Consequently, for sufficiently small $\varepsilon > 0$ the expression

$$g(y(t)) = g(x(t)) + \varepsilon \frac{\partial g(x(t))}{\partial x} \cdot \delta x(t) + o(\varepsilon), \quad t_1 \leq t \leq t_2$$

has a negative value. Indeed, for values of t close to t_1, this follows from the inequalities

$$g(x(t)) < 0, \quad a < 0.$$

For values of t far removed from t_1, the value of $|g(x(t))|$ is much greater that

$$\left|\varepsilon \frac{\partial g(x(t))}{\partial x} \cdot \delta x(t) + o(\varepsilon)\right|,$$

while

$$g(x(t)) < 0.$$

2. Basic definitions. Let $u(t), t_1 \leq t \leq t_2$ be a permissible control, and let $x(t)$ be the corresponding trajectory (not necessarily optimal) of equation (1.5) lying wholly in the closed region G. Some portions of the trajectory may lie on the boundary of region G, and some within the region, i.e., in the open kernel of region G.

Point $x(\tau)$ of the trajectory lying on the boundary of

region G will be called the <u>junction point</u> provided that $t_1 < \tau < t_2$ and there exists a $\sigma > 0$ such that at least one of the segments of trajectory $x(t)$ lies in the open kernel of region G for $\tau - \sigma < \tau$ or $\tau < t < \tau + \sigma$. Henceforth, for the sake of rigor, we shall always assume that the portion of the trajectory for which $\tau - \sigma < t < \tau$ lies within region G. Time τ will be called the <u>juncture moment</u>.

We shall consider trajectories with a finite number of junction points without mentioning this explicitly on each occasion.

Trajectory $x(t), t_1 \leq t \leq t_2$ which lies wholly in the closed region G will be called <u>regular</u> if all of its portions lying on boundary $g(x)=0$ of region G (see section 2, paragraph 1) are regular.

Let us assume that $u(t), t_1 \leq t \leq t_2$ is an optimal control, and that $x(t), t_1 \leq t \leq t_2$ is the corresponding optimal regular trajectory of equation lying wholly in region G.

Let $x(\tau)$ be a junction point of trajectory $x(t), t_1 \leq t \leq t_2$. Let us denote by $\tau_1 < \tau < \tau_2$ the maximum interval of segment $t_1 \leq t \leq t_2$ containing the single juncture moment τ.

Thus, a portion of trajectory $x(t)$ lies in the open kernel of region G for $\tau_1 < t < \tau$; as regards the segment for $\tau < t < \tau_2$, it lies wholly on boundary $g(x)=0$ or also belongs to the open kernel of region G, and then $x(\tau)$ is the only point of that segment of $x(t), \tau_1 < t < \tau_2$ which lies on the boundary of region G.

Consequently, the segment $x(t), \tau_1 \leq t \leq \tau$ satisfies the Maximum Principle. The non-zero function which corresponds to this segment,

$$\psi^-(t) = (\psi_0^-(t), \ldots, \psi_n^-(t)), \quad \tau_1 \leq t \leq \tau, \tag{3.8}$$

is continuous and satisfies system (3.1)-(3.3). Segment $x(t), \tau \leq t \leq \tau_2$ satisfies either the requirements of theorem 1 (if it lies on boundary $g(x)=0$), or the Maximum Principle (if it lies within region G). The corresponding continuous function

$$\psi^+(t) = (\psi_0^+(t), \ldots, \psi_n^+(t)), \quad \tau \leq t \leq \tau_2 \tag{3.9}$$

satisfies either system (2.14)-(2.16) and conditions a)-c) of theorem 1, or system (3.1)-(3.3).

We shall say that at junction point $x(\tau)$ of optimal regular trajectory $x(t), t_1 \leq t \leq t_2$ lying wholly in the closed region G the jump condition is satisfied if there exists a segment $x(t), \tau_1 \leq t \leq \tau_2$ of the trajectory such that $\tau_1 < t < \tau_2$ is the maximum interval of the segment $t_1 \leq t \leq t_2$ containing the single juncture moment τ, and if for the segments $x(t), \tau_1 \leq t \leq \tau, x(t), \tau \leq t \leq \tau_2$ the functions (3.8), (3.9) defined above can be so chosen that one the of the two following (incompatible) conditions is satisfied:

$$\psi^{+}(\tau) = \psi^{-}(\tau) + \mu \operatorname{grad} g(x(\tau)), \tag{3.10}$$

$$\psi^{-}(\tau) + \mu \operatorname{grad} g(x(\tau)) = 0, \quad \mu \neq 0, \tag{3.11}$$

where μ is a real number. If the segment $x(t), \tau \leq t \leq \tau_2$ lies on boundary $g(x) = 0$, then condition (3.10) is equivalent to the condition

$$\psi^{+}(\tau) = \psi^{-}(\tau),$$

since the initial value $\psi^{+}(\tau)$ of the function $\psi^{+}(t), \tau \leq t \leq \tau_2$ can be changed into a random vector of the form $\mu \operatorname{grad} g(x(\tau))$ (see note 4 following paragraph 1).

3. Theorem 2 (the jump condition). Let a regular optimal trajectory of equation (1.5) lying in the closed region G contain a finite number of junction points. Then the jump condition is satisfied at each junction point.

Proof. Let $u(t), t_1 \leq t \leq t_2$ be an optimal trajectory, $x(t)$ -- the corresponding optimal trajectory, $x(\tau)$ -- a junction point, and $\tau_1 < t < \tau_2$ -- the maximum interval containing the single juncture moment τ. For the sake of rigor we assume that the segment $x(t), \tau_1 < t < \tau$ lies on boundary $g(x) = 0$. Point $x(\tau_1)$ can either lie within region G or on its boundary. We shall assume at first that $x(\tau_1)$ lies within G; in this case, it is obvious that $\tau_1 = t_1$.

Let us introduce the equation

$$\dot{x}^{*} = -f(x^{*}, u^{*}) \tag{3.12}$$

and consider its solution on the segment $0 \leq t \leq \tau - \tau_1 + \varepsilon \rho^{*}$, where ρ^{*} is any real number and ε is a positive infinitely small magnitude. Obviously, the solution to (3.12) are the functions

$$u^*(t) = u(\tau - t),\ x^*(t) = x(\tau - t),\ 0 < t \le \tau - \tau + \varepsilon\rho^*. \tag{3.13}$$

Let us denote by K^* cone (3.5) with apex at the point $x(\tau_1) = x^*(\tau - \tau_1)$ constructed with the aid of trajectory (3.13) of equation (3.12). By K let us denote the cone (2.55) with apex at point $x(\tau_2)$ constructed with the aid of the regular portion of the optimal trajectory $x(t), \tau \le t \le \tau_2$. Cone K lies in the tangential surface $T(x(\tau_2))$ to boundary $g(x) = 0$ at point $x(\tau_2)$.

Cone K^* is formed by the vectors

$$x(\tau_1) + \hat{\delta}^*(\vartheta^*, \delta^* x(0), \rho^*), \tag{3.14}$$

where $\hat{\delta}^*$ is defined by formula (3.7). Cone K is formed by the vectors

$$x(\tau_2) + \hat{\delta}(\vartheta, R, \delta x(\tau), \rho, \delta u), \tag{3.15}$$

where $\hat{\delta}$ is given by formula (2.60). Vectors $\delta^* x(0), \delta x(t)$ satisfy the conditions:

$$\text{either } \delta^* x(0) = 0, \quad \text{or } \operatorname{grad} g(x(\tau)) \cdot \delta^* x(0) < 0, \tag{3.16}$$
$$\text{either } \delta x(\tau) = 0, \quad \text{or } \operatorname{grad} g(x(\tau)) \cdot \delta x(\tau) < 0.$$

In other words, the initial displacements $\varepsilon\delta^* x(0), \varepsilon\delta x(\tau)$ are either equal to zero or do not come in contact with boundary $g(x) = 0$ and are directed into region G.

In the following construction the vectors $\delta^* x(0), \delta x(\tau)$ involved in expressions (3.14), (3.15) are equal to each other:

$$\delta^* x(0) = \delta x(\tau). \tag{3.17}$$

To each set of parameters

$$\vartheta^*, \delta^* x(0), \rho^*, \vartheta, R, \delta x(\tau), \rho, \delta u \tag{3.18}$$

there corresponds a pair of points which are the ends of the corresponding varied trajectories

$$\begin{aligned} y^*(\tau - \tau_1 + \varepsilon\rho^*) &= x(\tau_1) + \hat{\delta}^* + o(\varepsilon), \\ y(\tau_2 + \varepsilon\rho) &= x(\tau_2) + \hat{\delta} + o(\varepsilon). \end{aligned} \tag{3.19}$$

It is easily seen (viz. the analogous argument in section 2) that with all of the permissible changes in the parameters (3.18) and condition (3.17) fulfilled, the pairs

$$(x(\tau_1)+\delta^*, x(\tau_2)+\delta)$$

form a convex cone L lying in the direct product $X^{n+1}\cdot X^{n+1}$, with apex at the point $(x(\tau_1), x(\tau_2))$:

$$L=\{(x(\tau_1), x(\tau_2))+(\delta^*, \delta)\}, (\delta^*x(0)=\delta x(\tau)). \quad (3.20)$$

The pairs of corresponding ends (3.19) form the same cone L with an accuracy of up to $o(\varepsilon)$.

Furthermore, it is clear that L is a subcone of $K^*\cdot K$ -- the direct product of cones K^*, K. Since $K\subset T(x(\tau_2))$,

$$L\subset K^*\cdot T(x(\tau_2)). \quad (3.21)$$

Finally, the following conclusions are also obvious:

$$L\supset k^*\cdot x(\tau_2), \quad L\supset x(\tau_1)\cdot k, \quad (3.22)$$

where the subcones $k^*\subset K^*$, $k\subset K$ are defined in paragraph 1 and paragraph 4 of section 2.

Let us denote by S the ray emerging from $x(\tau_2)$ and directed along the negative axis x^0. We shall show that the ray $x(\tau_1)\cdot S$ lying in the direct product $K^*\cdot T(x(\tau_2))$ is not an internal ray of cone L.

Let us assume the opposite. Then, just as in section 2, for any sufficiently small $\varepsilon>0$ it is possible to prove the existence of a permissible set of parameters (3.18) satisfying the condition (3.17) such that the pair of ends (3.19) of the corresponding varied trajectories

$$y^*(t),\ 0\leq t\leq\tau-\tau_1+\varepsilon\rho^*,\ y(t),\ \tau\leq t\leq\tau_2+\varepsilon\rho \quad (3.23)$$

lies on the ray $x(\tau_1)\cdot S$ and does not coincide with its initial point $(x(\tau_1), x(\tau_2))$:

$$y^*(\tau-\tau_1+\varepsilon\rho^*)=x(\tau_1),\ y(\tau_2+\varepsilon\rho)=x(\tau_2)+\eta(-1,0,\ldots,0);\ \eta>0. \quad (3.24)$$

From conditions (3.16), lemma 1, and the assumption that the point $x(\tau_1)$ is internal in G, it follows that for sufficiently small $\varepsilon>0$ trajectory $y^*(t)$, $0\leq t\leq\tau-\tau_1+\varepsilon\rho^*$ lies

wholly in G; moreover, trajectory $y(t), \tau \le t \le \tau_2 + \varepsilon\rho$ lies in G by construction (see section 2).

Let us define the permissible control $\tilde{u}(t), \tau_1 - \varepsilon\rho^* \le t \le \tau_2 + \varepsilon\rho$ and the corresponding trajectory $\tilde{x}(t), \tau_1 - \varepsilon\rho^* \le t \le \tau_2 + \varepsilon\rho$ of formula (1.5) by the following formulas:

$$\tilde{u}(t) = v^*(\tau - t),\ \tilde{x}(t) = y^*(\tau - t) \quad \text{for} \quad \tau_1 - \varepsilon\rho^* \le t \le \tau,$$

$$\tilde{u}(t) = v(t) \quad , \quad \tilde{x}(t) = y(t) \quad \text{for} \quad \tau \le t \le \tau_2 + \varepsilon\rho$$

where $y^*(t), y(t)$ are the varied trajectories (3.23) and $v^*(t), v(t)$ are the corresponding controls. Obviously, functions $\tilde{u}(t), \tilde{x}(t)$, $\tau_1 - \varepsilon\rho^* \le t \le \tau_2 + \varepsilon\rho$ satisfy equation (1.5), and, by condition (3.17 trajectory $\tilde{x}(t), \tau_1 - \varepsilon\rho^* \le t \le \tau_2 + \varepsilon\rho$ is continuous at point τ and, consequently, over the entire segment $\tau_1 - \varepsilon\rho^* \le t \le \tau_2 + \varepsilon\rho$. In addition to this, according to (3.24), we have:

$$\tilde{x}(\tau_1 - \varepsilon\rho^*) = x(\tau_1),\ \tilde{x}(\tau_2 + \varepsilon\rho) = (x^0(\tau_2) - \eta, x^1(\tau_2) \dots, x^n(\tau_2)),\ \eta$$

But these inequalities contradict the fact that the segment $x(t), \tau_1 \le t \le \tau_2$ of optimal trajectory $x(t), t_1 \le t \le t_2$ is likewise optimal.

Thus, ray $x(\tau_1) \cdot S$ is an internal ray for cone L. From inclusion (3.21) it follows that

$$\dim L \le 2n + 1$$

Consequently, there exists a 2n-dimensional support surface to L at apex $(x(\tau_1), x(\tau_2))$ lying in $X^{n+1} \cdot T(x(\tau_2))$ and separating cone L from ray $x(\tau_1) \cdot S$. Let us denote the vector which is orthogonal to this surface, lies in $X^{n+1} \cdot T(x(\tau_2))$ and is directed in the direction of ray $x(\tau_1) \cdot S$ as (χ^*, χ). We have

$$x(\tau_2) + \chi = x(\tau_2) + (\chi_0, \dots, \chi_n) \in T(x(\tau_2)), \quad (3.25)$$

$$(\chi^*, \chi) \cdot (\partial^*, \partial) = \chi^* \cdot \partial^* + \chi \cdot \partial \le 0,\ \chi_0 \le 0 \quad (3.26)$$

where ∂^*, ∂ are defined by formulas (3.14), (3.15) under the condition (3.17).

The vector

$$(\chi^*, \chi) \ne 0 \quad (3.27)$$

and for this reason the vectors χ^*, χ do not simultaneously go to zero.

Let us denote by

$$\psi^*(t)\ ,\quad 0 \le t \le \tau - \tau_1 \qquad (3.28)$$

the solution to the equation

$$\dot\psi^* = \frac{\partial f(x^*(t), u^*(t))}{\partial x}\psi^*,$$

which satisfies the boundary condition

$$\psi^*(\tau-\tau_1) = \chi^*,$$

where the functions $u^*(t), x^*(t)$ are defined by formula (3.13). By

$$\psi^+(t),\quad \tau \le t \le \tau_2 \qquad (3.29)$$

let us denote the solution to the equation

$$\dot\psi = -\frac{\partial f(x(t), u(t))}{\partial x}\psi$$

with the boundary condition

$$\psi(\tau_2) = \chi.$$

Using inclusions (3.22) as in paragraph 1 and in section 2 we obtain:

$$-\psi^*(t)\cdot f(x^*(t), u^*(t)) = H(-\psi^*(t), x^*(t), u^*(t)) =$$

$$= M(-\psi^*(t), x^*(t)) = 0\ ,\quad 0 \le t \le \tau - \tau_1,$$

$$H(\psi^+(t), x(t), u(t)) = m(\psi^+(t), x(t)) = 0,\ \tau \le t \le \tau_2. \qquad (3.30)$$

In addition to the latter equation, the function $\psi^+(t)$, $\tau \le t \le \tau_2$ satisfies all conditions of theorem 1, with the possible exception of condition b), since the equality $\chi = 0$, and, consequently the identity $\psi^+(t) \equiv 0$ are not excluded.

Let

$$\psi^-(t) = -\psi^*(\tau - t),\ \tau_1 \le t \le \tau. \qquad (3.31)$$

Obviously,

$$\dot\psi^-(t) = -\frac{\partial f(x(t), u(t))}{\partial x}\psi^-(t),\ \psi^-(\tau_1) = -\chi^*;\ \tau_1 \le t \le \tau,$$

$$H(\psi^-(t), x(t), u(t)) = M(\psi^-(t), x(t)) = 0$$

From inequality (3.27) it follows that at least one of the functions $\psi^-(t), \psi^+(t)$ is different from zero. It is proved below that $\psi^-(t) \ne 0$ at all times, and that, consequently,

$\psi^-(\tau)$ satisfies all the conditions of the Maximum Principle.

Let us now prove that either

$$\psi^+(\tau) = \psi^-(\tau) + \mu \operatorname{grad} g(x(\tau)), \quad \psi^-(t) \neq 0$$

or

$$\psi^-(\tau) + \mu \operatorname{grad} g(x(\tau)) = 0, \quad \mu \neq 0$$

Thereby the jump conditions will be proved, since function $\psi^-(t)$, $\tau_1 \leq t \leq \tau$ can be taken for function (3.8), and function $\psi^+(t)$, $\tau \leq t \leq \tau_2$ for function (3.9), if $\psi^+(t) \neq 0$.

The varied trajectories whose ends are given by formula (3.19) will be chosen in the following manner.

Let us set

$$\delta^* = \delta = 0, \quad \delta u = 1, \quad v^*(t) = u^*(t),$$

$$\delta^* x(\tau) = \delta x(\tau) = -N,$$

where N is a random vector not tangential to boundary $g(x)=0$ at point $x(\tau)$ and directed out from region G; randomicity in the choice of N is achieved through the appropriate selection of function R (see section 2, paragraph 4). Finally, control v will be chosen without defining points, so that trajectory $y(t)$ will be constructed on the basis of the selected parameters $\rho = 0$, $\delta u = 1$, $\delta x(\tau) = -N$, R with the aid of the construction used in proving lemma 1 of section 2. For vectors δ^*, δ defined according to formulas (3.19), we obtain the expressions (see (3.7),(2.43)):

$$\delta^* = -\sum_{\alpha=0}^{n} \varphi_\alpha^* (\tau - \tau_1) N_*^\alpha \quad ; \tag{3.32}$$

$$\delta = \sum_{\alpha=0}^{n} \varphi_\alpha (\tau_2) \left(-N^\alpha + \int_{\tau}^{\tau_2} \psi^\alpha(t) \cdot \Lambda(t) \frac{\partial R}{\partial u} dt\right),$$

where

$$\sum_{\alpha=0}^{n} \varphi_\alpha^* (0) N_*^\alpha = \sum_{\alpha=0}^{n} \varphi_\alpha (\tau) N^\alpha = N$$

Here $\varphi_0^*(t), \ldots, \varphi_n^*(t)$, $0 \leq t \leq \tau - \tau_1$ denotes the fundamental system of solutions to the equation in variations for equation (3.12).

By $\psi_*^0(t) \ldots \psi_*^n(t)$ let us denote the system of functions conjugate to $\varphi_0^*(t), \ldots, \varphi_n^*(t)$. We have:

$$-\chi^{*}\cdot\delta^{*}-\sum_{\alpha=0}^{n}\chi^{*}\cdot\varphi_{\alpha}^{*}(\tau-\tau_{1})N_{*}^{\alpha}=$$

$$=\sum_{\alpha,\beta=0}^{n}(\chi_{\beta}^{*}\psi_{*}^{\beta}(\tau-\tau_{1})\cdot(\varphi_{\alpha}^{*}(\tau-\tau_{1})N_{*}^{\alpha})=\sum_{\alpha=0}^{n}\chi_{\alpha}^{*}N_{*}^{\alpha}=\psi^{*}(0)\cdot N,$$

where $\psi^{*}(0)$ is the initial value of function (3.29). From equation (3.31) follows:

$$\chi^{*}\cdot\delta^{*}=\psi^{-}(\tau)\cdot N. \tag{3.33}$$

Analogously, the second equation of (3.32) yields:

$$\chi\cdot\delta=-\psi^{+}(\tau)\cdot N+\int_{\tau}^{\tau_2}\psi^{+}(t)\cdot\Lambda(t)\frac{\partial R}{\partial\mu}\,dt=-\psi^{+}(\tau)\cdot N-\int_{\tau}^{\tau_2}\lambda(t)\frac{\partial R}{\partial\mu}\,dt,$$

where $\lambda(t)=-\psi^{+}(t)\cdot\Lambda(t)$ Using expression (2.44) for $\frac{\partial R}{\partial\mu}$ and integrating by parts, we obtain:

$$\chi\cdot\delta=-\psi^{+}(\tau)\cdot N-\left[\lambda(t)\sum_{i=1}^{s}a_i(x(t))\frac{\partial g(x(t))}{\partial x}\cdot N_i\right]\Bigg|_{\tau}^{\tau_2}+$$

$$+\int_{\tau}^{\tau_2}\lambda'\sum_{i=1}^{s}a_i(x(t))\frac{\partial g(x(t))}{\partial x}\cdot N_i\,dt=$$

$$=-\psi^{+}(\tau)\cdot N+\lambda(t)\frac{\partial g(x(\tau))}{\partial x}\cdot N+\int_{\tau}^{\tau_2}\lambda'\sum_{i=1}^{s}a_i(x(t))\frac{\partial g(x(t))}{\partial x}\cdot N_i\,dt.$$

Adding this equation to (3.33), and taking into account equation (3.26), we find:

$$\chi^{*}\cdot\delta^{*}+\chi\delta=\left(\psi^{-}(\tau)-\psi^{+}(\tau)+\lambda(\tau)\frac{\partial g(x(\tau))}{\partial x}\right)\cdot N+$$

$$+\int_{\tau}^{\tau_2}\lambda'\sum_{i=1}^{s}a_i(x(t))\frac{\partial g(x(t))}{\partial x}\cdot N_i\,dt\le 0. \tag{3.34}$$

The value of $\int_{\tau}^{\tau_2}\lambda'\sum_{i=1}^{s}a_i(x(t))\frac{\partial g(x(t))}{\partial x}\cdot N_i\,dt$

can be made arbitrarily small with any given N through the selection of small neighborhoods $\mathcal{E}_{\xi_i}$ entering into the definition of function R, while the first term of equation (3.34) is independent of these neighborhoods. Consequently, for random vector N which is not tangent to boundary $g(x)=0$ at point $x(\tau)$ and is directed out from region G, there holds

the inequality

$$[\psi^-(\tau) - \psi^+(\tau) + \lambda(\tau)\,\mathrm{grad}\, g(x(\tau))] \cdot N \leq 0,$$

which, due to the randomicity of vector N, is equivalent to the equation

$$\psi^+(\tau) = \psi^-(\tau) + \mu\,\mathrm{grad}\, g(x(\tau)). \tag{3.35}$$

Vector $\psi^-(\tau) \neq 0$, since from equation $\psi^-(\tau) = 0$ and inequality (3.27) there follows the inequality $\psi^+(\tau) \neq 0$; on the other hand, from (3.35) we obtain the relationship

$$\psi^+(\tau) = \mu\,\mathrm{grad}\, g(x(\tau)) \neq 0$$

which contradicts the inclusion (3.25).

With $\psi^+(\tau) = 0$ we obtain:

$$\psi^-(\tau) = \mu\,\mathrm{grad}\, g(x(\tau)) = 0 \quad , \mu \neq 0$$

Thus, theorem 2 is proved for the case where $x(\tau_1)$ is an internal point in region G.

Now let $x(\tau_1)$ lie on the boundary $g(x)=0$. This case is easily reduced to the one just considered: it is sufficient to define function $\psi_\theta^-(t)$ on segment $\theta \leq t \leq \tau$, where $\tau_1 < \theta$, and then to make point θ go to τ_1 ; we thus obtain a family of functions $\psi_\theta^-(t), \theta \leq t \leq \tau$ for which there exists the required limiting function $\psi^-(t), \tau_1 \leq t \leq \tau$.

Note 1. If the segment $x(t), \tau < t < \tau_2$ also belongs to the open kernel of region G, the inequality (3.34) will be replaced by the inequality

$$(\psi^-(\tau) - \psi^+(\tau)) \cdot N \leq 0,$$

whence follows the equation

$$\psi^+(\tau) = \psi^-(\tau) + \mu\,\mathrm{grad}\, g(x(\tau)), \quad \mu \geq 0.$$

Note 2. If the segment $x(t), \tau \leq t \leq \tau_2$ lies on boundary $g(x) = 0$, vectors $\mathrm{grad}\, g(x(\tau))$ and $f(x(\tau), u(\tau+0))$ are orthogonal and the jump condition yields:

$$\psi^-(\tau) \cdot f(x(\tau), u(\tau-0)) = (\psi^-(\tau) + \mu\,\mathrm{grad}\, g(x(\tau))) \cdot f(x(\tau), u(\tau+0)) =$$
$$= \psi^-(\tau) \cdot f(x(\tau), u(\tau+0)) = M(\psi^-(\tau), x(\tau)) = 0.$$

From this it follows that if the system of equations with respect to u:

$$\psi^-(\tau) \cdot f(x(\tau), u) = M(\psi^-(\tau), x(\tau)) = 0$$

has the single solution

$$u(\tau-0) = u(\tau+0)$$

then the vector

$$f(x(\tau), u(\tau-0)) = f(x(\tau), u(\tau+0))$$

touches boundary $g(x)=0$ at point $x(\tau)$; in other words, the optimal trajectory remains smooth at the junction point $x(\tau)$.

Note 3. If the optimal trajectory lies on the piecewise smooth boundary of region G (we have heretofore been considering region G with a smooth boundary), then the equations of motion of any segment of the trajectory lying wholly on the smooth portion of the boundary have already been found in section 2. As the trajectory passes from one smooth portion of the boundary to the next, jump conditions completely analogous to (3.10), (3.11) are fulfilled.

Net 4. The jump conditions are also applicable to the following optimal problem.

Let the phase space X^{n+1} be divided into two parts X_1, X_2 by the hypersurface $g(x)=0$. In part X_1, let the equation of mation of the phase point have the form

$$\dot{x} = f_1(x, u),$$

and in X_2 -- the form

$$\dot{x} = f_2(x, u).$$

We must select a permissible control such that the phase point passes form the initial position $\xi \in X_1$ onto the straight line $\pi \subset X_2$ parallel to axis x^0, and the coordinate x^0 of the end of the trajectory goes to a minimum.

The trajectory of motion in each of the parts X_1, X_2 will satisfy the Maximum Principle, while the jump condition will be fulfilled at the moment of psage across the boundary $g(x)=0$.

In the derivation of the jump condition in the present case, the initial displacement of the varied trajectory must lie strictly on the boundary $g(x)=0$ dividing the two spaces. For this reason, the lemma of paragraph 1 cannot be used and the proof of section 3 is applicable only if not one of the vectors $f(x(\tau), u(\tau-0)), f(x(\tau), u(\tau+0))$ touches hypersurface $g(x)=0$ at junction point $x(\tau)$. I do not know whether the jump condition holds when at least one of these vectors touches the boundary $g(x)=0$ dividing the two spaces.

If we are concerned with a standard variational problem, the known conditions of extremal refraction. As an example, we will now derive these conditions for a very

simple variational problem.

Let the plane x, y be divided by the line $g(x,y)$ into two parts X_1, X_2, and let there be given two points $(x_1, y_1) \in X_1$ $(x_2, y_2) \in X_2$. We are required to connect these points with a continuous piecewise-smooth line $y = y(x)$ in such a way that the integral

$$\int_{x_1}^{x_2} F(x, y, y')\, dx,$$

where

$$F(x, y, y') = f_1(x, y, y') \text{ for } (x, y) \in X_1,$$
$$F(x, y, y') = f_2(x, y, y') \text{ for } (x, y) \in X_2$$

and f_1, f_2 are smooth functions of their arguments, assumes a minimum value.

Let us introduce the definitions:

$$x^0 = \int_{x_1}^{x} F(x, y, y')\, dx \quad , \quad x^1 = x, \; x^2 = y, \; u = y'.$$

The region of possible values for the control is the open set of a numerical straight line. The Maximum Principle is written down in the following manner (it is easy to show that $\psi_0 \neq 0$ and consequently that it is possible to set $\psi_0 = -1$):

$$\dot{x}^0 = F(x, y, y') \quad , \quad \dot{\psi}_0 = 0$$
$$\dot{x}^1 = 1, \qquad \dot{\psi}_1 = \frac{\partial F}{\partial x},$$
$$\dot{x}^2 = u = y', \qquad \dot{\psi}_2 = \frac{\partial F}{\partial y}$$

$$H = -F(x, y, y') + \psi_1 + \psi_2 y' = \max = 0.$$

The conditions H = max and H = 0 give, respectively:

$$\psi_2 = \frac{\partial F}{\partial y'} \quad , \quad \psi_1 = F - \frac{\partial F}{\partial y'} y'.$$

From the jump condition $\psi^+ = \psi^- + \mu \operatorname{grad} g(x,y)$ there follows:

$$\psi_1^+ = f_2 - \frac{\partial f_2}{\partial y'} (y^+)' = f_1 - \frac{\partial f_1}{\partial y'} (y^-)' + \mu N^1,$$
$$\psi_2^+ = \frac{\partial f_2}{\partial y'} = \frac{\partial f_1}{\partial y'} + \mu N^2,$$

where (N^1, N^2) is the normal vector to the line $g(x,y)$ at the point of fracture of the trajectory. Let us denote by γ' the angle of declination of the tangent to the curve

$g=0$ at the point of fracture. We have:

$$\frac{\frac{\partial f_2}{\partial y'} - \frac{\partial f_1}{\partial y'}}{f_2 - f_2 + \frac{\partial f_1}{\partial y'}(y^-)' - \frac{\partial f_2}{\partial y'}(y^+)'} = -\frac{1}{Y'}$$

whence we obtain the known formula (see ref. 10):

$$f_1 + \frac{\partial f_1}{\partial y'}\left(Y' - (y^-)'\right) = f_2 + \frac{\partial f_2}{\partial y'}\left(Y' - (y^+)'\right).$$

4. General Principle for Determining Optimal Trajectories

Combining theorems 1 and 2 and the Maximum Principle, we arrive at the following theorem, which gives the full system of necessary conditions to be satisfied by any regular optimal trajectory that is a solution of the optimal problem in section 1.

Theorem 3. Let the optimal trajectory of equation (1.5) lie entirely in the closed region G and contain a finite number of junction points; furthermore, let any portion of this trajectory lying on the boundary of region G be regular. Then any portion of the trajectory lying in the open kernel of region G (with the possible exception of the ends of the trajectory) satisfies the Maximum Principle; any of its portions which lie on the boundary of region G satisfy theorem 1; the jump condition (theorem 2) is fulfilled at every junction point.

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